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Metric Dimension of Banded-Turán Graph

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ABSTRACT

Given a graph $G = \big(V(G), E(G)\big)$. Let $S = \{s_1, \dots, s_k\}$ be an ordered subset of $V(G)$. Consider a vertex $x \in V(G)$, a coordinate of x with respect to S is represented as $r(x|S) = \big(d(x, s_1), \dots, d(x, s_k)\big)$ where $d(x, s_i)$ equals the number of edges in the shortest path between x and s_i for $i = 1, \dots, k$. The minimum value of k such that for every $x \in V(G)$ has distinct coordinate is called metric dimension of G, denoted by $\beta(G)$. Turán graph $T(n, r), n \geq r \geq 2$, is a subgraph of complete r-partite graph on n vertices having property that the difference of cardinality of any two distinct classes is at most one. In this paper, we build a new graph namely a banded-Turán graph, $BT(n, r, m)$, as a graph built by a Turán graph $T(n, r)$ and r uniform path graphs P_m in which every vertex of each class of $T(n, r)$ connected to an initial vertex of corresponding path graph P_m. Intuitively, this graph illustrates as if a Turán graph is banded by r uniform ropes. We determine some basic graph properties including independence number, chromatic number, and diameter of banded-Turán. The main result in this paper is that the metric dimension of banded-Turán turns out that it depends on the number of its classes. If it has two classes then the metric dimension is equal to $n - 1$ and if it has more than two classes then the metric dimension is equal to the metric dimension of Turán graph included in it.

Diberikan sebuah graf $G = \big(V(G), E(G)\big)$. Misalkan himpunan $S = \{s_1, \dots, s_k\}$ adalah subset terurut dari $V(G)$. Diperhatikan sebuah titik $x \in V(G)$, koordinat dari x relatif terhadap S direpresentasikan sebagai $r(x|S) = \big(d(x, s_1), \dots, d(x, s_k)\big)$ dengan $d(x, s_i)$ menyatakan banyaknya sisi terpendek antara x dan s_i untuk $i = 1, \dots, k$. Nilai minimum k sedemikian sehingga untuk setiap $x \in V(G)$ memiliki koordinat yang berbeda disebut sebagai dimensi metrik dari graf G, dinotasikan dengan $\beta(G)$. Graf Turán $T(n, r), n \geq r \geq 2$, adalah suatu subgraf dari graf r-partit lengkap dengan n titik yang mempunyai sifat bahwa selisih kardinalitas sebarang dua kelasnya paling besar 1. Dalam paper ini, kami membangun sebuah graf baru yang diberi nama banded-Turán graph, $BT(n, r, m)$, sebagai suatu graf yang dibangun oleh graf Turán $T(n, r)$ dan r graf lintasan seragam P_m yang mana setiap titik di masing-masing kelas dari $T(n, r)$ terhubung pada sebuah titik pangkal dari graf lintasan P_m yang bersesuaian. Secara intuitif, graf ini mengilustrasikan seolah-olah suatu graf Turán diikat oleh sebanyak r tali yang seragam. Kami menentukan beberapa sifat dasar graf yang meliputi bilangan independensi, bilangan kromatik, dan diameter dari graf banded-Turán. Hasil utama dalam tulisan ini adalah dimensi metrik dari graf banded-Turán yang ternyata bergantung pada banyak kelasnya. Jika graf tersebut memiliki dua kelas maka dimensi metriknya sama dengan $n - 1$ dan jika graf tersebut memiliki lebih dari dua kelas maka dimensi metriknya sama dengan dimensi metrik dari graf Turán yang termuat didalamnya.

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INTRODUCTION

Graph theory was first studied by a Swiss mathematician, Leonhard Euler, in 1735 while he was solving a seven Königsberg Bridge Problem, where next year the result published in Proceeding Petersburg Academy under the title "Solutio Problematis ad Geometriam Situs Pertinentis" (Alexanderson, 2006). In modern era, we see a lot of applications of graph theory such as used in timetable scheduling (Basudeb & Kajal, 2017), chemical molecules & ecosystem (Gross et al., 2019), communication network (call graphs) & transportation network (Rosen, 2012), and so forth.

Graf G is an unordered pair of a finite nonempty set of vertices $V(G) = \{v_1, \ldots, v_n\}$ and a finite set of edges $E(G) = \{e_1, \ldots, e_m\}$. A vertex coloring of a graph is an assignment of colors to the vertices so that adjacent vertices have distinct colors (Rahman, 2017). A chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that a graph G has k colors. The graph coloring has many applications in real life such as coloring map (Rahman, 2017) and solving Sudoku problem (Adrianto et al., 2015). The independent set I of G is the subset of $V(G)$ such that a graph induced by I has no edges. The independent set of G which has maximal cardinality is called a maximal independent set of G and its cardinality specifically called an independence number of G, denoted by $\alpha(G)$. The sequence of vertices and edges of G i.e. $x_1, e_1, x_2, e_2, \ldots, e_m, x_{m+1}$ such that it has no x_i, x_j with $x_i = x_j$ for all $i \neq j$ is called a $x_1 - x_{m+1}$ path in G. The distance between two distinct vertices x and y of G, denoted by $d(x, y)$, is the length of a shortest $x - y$ path in G. If x and y are adjacent to each other then it is denoted by $x \sim y$. The diameter of G is the maximal distance between any two distinct vertices in G, denoted by $\operatorname{diam}(G)$. Let $S = \{s_1, s_2, \ldots, s_k\}$ be an ordered subset of $V(G)$. For $x \in V(G)$, a coordinate of x with respect to S is defined as k-entry of the shortest length of x to the element of S i.e.

$$r(x|S) = \big(d(x, s_1), d(x, s_2), \ldots, d(x, s_k)\big).$$

The set S is a resolving set of G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x|S) \neq r(y|S)$. Khuller et al. (1996) stated that the elements of S are called *landmarks*. A basis of G is a resolving set of G with minimum cardinality, and the metric dimension of G refers to its cardinality, denoted by $\beta(G)$. The metric dimension in general graphs was first studied by Harary and Melter (1976) and independently by Slater (1975, 1988). Khuller et al. (1996) and Cáceres et al. (2010) showed that determining the metric dimension of graph is NP-hard problem. Some result for certain class of graphs have been obtained, such as paths (Chartrand et al., 2000; Khuller et al., 1996) and its complements (Alholi et al., 2017), regular bipartites (Saputro et al., 2011), complete graphs (Chartrand et al., 2000), Turán graphs $T(m^2, m)$ (Mutianingsih, 2016), trees (Chartrand et al., 2000; Khuller et al., 1996), grids (Melter & Tomescu, 1984), generalized Petersen graphs (Shao et al., 2018), Cayley digraphs (Fehr et al., 2006), stars (Chartrand et al., 2000), complete n-partite graphs (Saputro et al., 2009), corona product of graphs (Iswadi et al., 2011), arithmetic graph of a composite number (Rehman et al., 2020), total graph of a finite commutative ring (Dolzan, 2016), fullerene graphs (Akhter & Farooq, 2019), circulant graphs (Imran et al., 2012), and random graphs (Bollobas et al., 2013). On the other hand, there is some research applying metric dimension concepts such as determining robots navigation (Khuller et al., 1994), solving coin weighing problems and combinatorial search and optimization (Sebo & Tannier, 2004), metric dimension of sparse graph to prove a conjecture of Eroh *et al* (Bosquet et al., 2021), pattern recognition and image processing (Khuller et al., 1994), and pharmaceutical chemistry (Chartrand et al., 2000). In schools, it possibly realizes that a chemical compound or materials can be represented by a labelled graph whose vertex and edge labels specify the atom and bond types, respectively. Furthermore, a graph theoretic interpretation provides coordinates for the vertices of a graph in such a way that distinct vertices have distinct coordinates (Zhang & Naeem, 2021).

Let $\mathcal{H} = \{V_1, \ldots, V_r\}$ be a partition of $V(G)$ such that V_i be an independent set of G for each i. Those independent sets ultimately called as classes of G. A graph $K_{|V_1|,\ldots,|V_r|}$ is called a complete r-partite graph if and only if it has such $\mathcal{H}$ and any two vertices $x \in V_i$ and $y \in V_j$ for all $i, j \in \{1, \ldots, r\}$ satisfy $x \sim y$. Turán graph $T(n, r)$ is a subgraph of complete r-partite graph on n vertices having property that the difference of cardinality of any two distinct classes is at most one. The following theorem is called Turán's theorem, states the number of edges of graph G contains no complete subgraph K_{r+1}.

Theorem 1.1 (Aigner, 1995) *Let $n \geq r$ be positive integers. If G be a graph with n vertices without complete subgraph K_{r+1} then G has at most $\frac{(r-1)n^2}{2r}$ edges.*

We have the following corollary for Turán graph.

Corollary 1.1 *Let $t(n,r)$ be the number of edges of $T(n,r)$ for $n \geq r \geq 2$. Then $t(n,r) = \left\lfloor \frac{(r-1)n^2}{2r} \right\rfloor$.*

In this paper, we build a new graph that is called banded-Turán graph and discuss its graph properties including the independence number, chromatic number, diameter, and particularly its metric dimension.

RESULTS

A formal definition of banded-Turán graph where the name comes by looking the shape of the graph that can be viewed as a Turán graph banded by some uniform ropes such that every rope connects to one class in certain way.

Definition 2.1 *Given a Turán graph $T(n,r)$ with its partition is $\{V_1, V_2, \dots, V_r\}$ and path graphs $P_{(i,m)}$ for all $1 \leq i \leq r$, $n \geq r \geq 2$, and $m \in \mathbb{N}$. Let $\varepsilon_{(i,1)}, \dots, \varepsilon_{(i,m)} \in V(P_{(i,m)})$, $\lambda_{(i,j)} \in V_i$, and $|V_i| = n_i$, for every $1 \leq j \leq n_i$. Define a banded-Turán graph $BT(n,r,m)$ as a graph formed by Turán graph $T(n,r)$ and path graphs $P_{(i,m)}$ in a way that vertices $\lambda_{(i,j)}$ and $\varepsilon_{(i,1)}$ are joined by edge $e_{(i,j)}$ i.e. $e_{i,j} = \lambda_{i,j}\varepsilon_{(i,1)}$ for each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n_i\}$.*

See the following figure as an example of banded-Turán graph $BT(n,r,n)$.

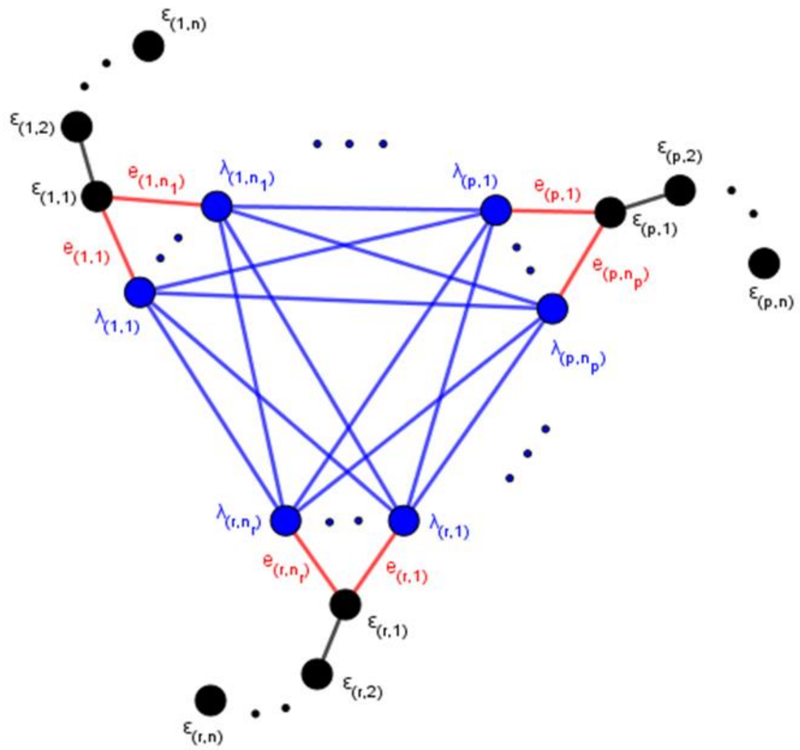

Figure 1. $BT(n,r,n)$ for $n \geq r \geq 2$ and $1 < p < r$

In the figure 1, Turán graph $T(n,r)$ and path graphs $P_{(i,n)}$ $(1 \leq i \leq r)$ are described by blue graph and black graph, respectively. Moreover, red edges describe the edges joining end vertices of each path to all vertices in each class of $T(n,r)$. Banded-Turán graph is always connected in any given possible cases since Turán graph has been connected and powered by the definition of $BT(n,r,m)$. Next, we consider the following fact about graph $BT(n,r,m)$.

Proposition 2.2 *Let $n \geq r \geq 2$, $1 \leq i \leq r$, $1 \leq j \leq n_i$, and $1 \leq k \leq m$. Given $BT(n,r,m)$ with $\lambda_{(i,j)} \in V_i$ and $\varepsilon_{(i,k)} \in V(P_{(i,m)})$, then the following statements are true:*

i. $d(\lambda_{(i,j)}, \varepsilon_{(i,k)}) = k$

ii. $d\left(\varepsilon_{(i,k)}, v_0\right) - d\left(\lambda_{(i,j)}, v_0\right) = k,\ v_0 \in V\left(BT(n,r,m)\right) \backslash \left(V_i \cup V\left(P_{(i,m)}\right)\right)$.

Proof. For all $1 \leq i \leq r, 1 \leq j \leq n_i$, and $1 \leq k \leq m$, see the following figure.

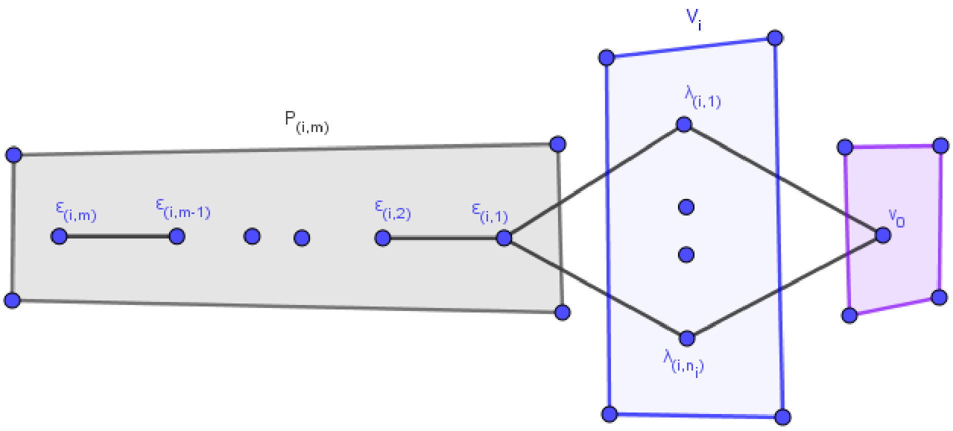

Figure 2. Graph G_1

Graph G_1 in figure 2, shows that $\varepsilon_{(i,1)} \sim \lambda_{(i,j)}$, $\varepsilon_{(i,2)} \sim \varepsilon_{(i,1)}$, and $\varepsilon_{(i,2)} \nsim \lambda_{(i,j)}$. These imply that $d\left(\varepsilon_{(i,1)}, \lambda_{(i,j)}\right) = 1$ and $d\left(\varepsilon_{(i,2)}, \lambda_{(i,j)}\right) = 2$. Inductively, we obtain $d\left(\varepsilon_{(i,k)}, \lambda_{(i,j)}\right) = k$. Moreover, we consider that $\lambda_{(i,j)} \sim v_0$ where $v_0 \in V\left(BT(n,r,m)\right) \backslash \left(V_i \cup V\left(P_{(i,m)}\right)\right)$, so that we obtain $d\left(\lambda_{(i,j)}, v_0\right) = 1$. Since $d\left(\varepsilon_{(i,k)}, \lambda_{(i,j)}\right) = k$, then $d\left(\varepsilon_{(i,k)}, v_0\right) = k + 1$. Therefore, $d\left(\varepsilon_{(i,k)}, v_0\right) - d\left(\lambda_{(i,j)}, v_0\right) = k$. ■

Now, we will discuss some characterizations of banded-Turán graph. The following proposition provides a chromatic number of path and Turán graph.

Proposition 2.3 *For $n \geq r \geq 2$, then $\chi(P_n) = 2$ and $\chi\left(T(n,r)\right) = r$.*

Proof. Let $v_1, v_2, \ldots, v_n \in V(P_n)$, and i and j are distinct positive integers less than or equal to n. We consider that if i and j are both odd or even then $v_i \nsim v_j$. So that v_i and v_j can have the same color. However, if i is odd and j is even (or vise versa) then v_i and v_j can not have the same color since both are adjacent to each other. Hence, we only need two colors to assign vertices of path graph for all n i.e., $\chi(P_n) = 2$. Next, we consider that all vertices in each class of $T(n,r)$ are not adjacent, so they can have the same color. Nevertheless, every class must has a unique color since all vertices in one class adjacent to all vertices in other classes. Hence, the chromatic number of $T(n,r)$ is equal to the number of its class i.e., $\chi\left(T(n,r)\right) = r$. ■

Furthermore, the following lemma is a useful tool to determine a metric dimension of some graphs.

Lemma 2.4 *Let S be a resolving set of G. If there exists a set $H \subset V(G)$ such that all elements of H have the same distance to $x \in V(G) \backslash H$, then $|H| - 1$ vertices of H must become landmarks in S.*

Proof. Let $H = \{u_1, u_2, \ldots, u_r\}$. Since we know that $d(u_1, x) = d(u_2, x) = \cdots = d(u_r, x)$ for every $x \in V(G) \backslash H$, then all elements of H can not be distinguished by x. In order to make S as a resolving set of G, then it must be $r - 1$ vertices of H become landmarks in S. ■

The following proposition provides the metric dimension of Turán graph which the proof used a simple argument.

Proposition 2.5 *Let $\delta \geq 0$ be the number of classes in $T(n,r)$ containing exactly one vertex for $n \geq r \geq 2$. Then*

$$\beta\left(T(n,r)\right) = \begin{cases} r - 1, & \delta > 0 \\ n - r, & \delta = 0. \end{cases}$$

Proof. Let $S \subseteq V\left(T(n,r)\right)$ be a basis of $T(n,r)$. The first case, let $\delta > 0$. Then, there are some classes with cardinality of $\lfloor n/r \rfloor = 1$ or $\lceil n/r \rceil = 2$. If there are δ classes having exactly one vertex in $T(n,r)$ then by using Lemma 2.4, we obtain the number of landmarks in S is $(r - \delta) + (\delta - 1) = r - 1$. The second case, let $\delta = 0$. Then, every class has the cardinality of at least two i.e., $\lceil n/r \rceil \geq \lfloor n/r \rfloor > 1$. If there are x class with cardinality of

$\lceil n/r \rceil$. Using Lemma 2.4, we obtain the number of landmarks in S is $x(\lceil n/r \rceil - 1) + (r - x)(\lfloor n/r \rfloor - 1) = n - r$. ■

We have the following proposition provides the number of edges of banded-Turán graph using Corollary 1.1.

Proposition 2.6 *For* $n \geq r \geq 2$ *and* $m \in \mathbb{N}$*, then* $\left|E\big(BT(n,r,m)\big)\right| = t(n,r) + n + r(m-1)$.

Proof. Based on Definition 2.1, since every class contains $e_{(i,j)}$ where $1 \leq i \leq r$ and $1 \leq j \leq n_r$, then the number of $e_{(i,j)}$ equals n. Moreover, it's easy to see that a path graph of order m has $m-1$ edges and according to Corollary 1.1, $T(n,r)$ has $t(n,r)$ edges. Since $BT(n,r,m)$ contains exactly one $T(n,r)$ and r uniform path graphs of m vertices, then we obtain the number of edges of $BT(n,r,m)$ is $t(n,r) + n + r(m-1)$ for $n \geq r \geq 2$ and $m \in \mathbb{N}$. ■

The following proposition provides the independence number of banded-Turán graph.

Proposition 2.7 *For* $n \geq r \geq 2$ *and* $m \in \mathbb{N}$*, then*

$$\alpha\big(BT(n,r,m)\big) = \max\{|V_i| : 1 \leq i \leq r\} + (r-2)\left\lceil \frac{m}{2} \right\rceil + m.$$

Proof. Consider a Turán graph $T(n,r)$ in $BT(n,r,m)$. Since any two vertices of distinct classes are adjacent and any two vertices of the same class are not adjacent then it easy to know that the maximal independent set in $T(n,r)$ is the class with maximal number of vertices i.e. $\alpha\big(T(n,r)\big) = \max\{|V_i| : 1 \leq i \leq r\}$. Next, for the path graph $P_{(i,m)}$ that is banding the maximal independent set of $T(n,r)$, the maximal independent set must have $\left\lfloor \frac{m}{2} \right\rfloor$ independence number. For the rest $r-1$ path graphs are disjoint with the maximal independent set, so they have the independence number $\left\lceil \frac{m}{2} \right\rceil$. Therefore, we obtain the independence number of banded-Turán graph is $BT(n,r,m) = \max\{|V_i| : 1 \leq i \leq r\} + (r-2)\left\lceil \frac{m}{2} \right\rceil + m$. ■

Using Proposition 2.3, we have the following proposition shows that the number of colors to color Turán graph is sufficient to color banded-Turán graph.

Proposition 2.8 *For* $n \geq r \geq 2$ *and* $m \in \mathbb{N}$*, then* $\chi\big(BT(n,r,m)\big) = \chi\big(T(n,r)\big)$.

Proof. Since $BT(n,r,m)$ contains $T(n,r)$ and $P_{(i,m)}$ $(1 \leq i \leq r)$, then based on Proposition 2.3, we obtain a bound of chromatic number of $BT(n,r,m)$ as follows:

$$r \leq \chi\big(BT(n,r,m)\big) \leq r + 2.$$

We claim that $\chi\big(BT(n,r,m)\big)$ equals its lower bound. Let $\{w_i\}$ be a set of colors which one to one correspondence to the set $V_i \subset V(BT(n,r,m)$ for each $1 \leq i \leq r$. By the Definition 2.1, since $\lambda_{(i,j)} \sim \varepsilon_{(i,1)}$, then $\lambda_{(i,j)}$ and $\varepsilon_{(i,1)}$ can not have the same color. Furthermore, the result for all $n \geq r \geq 2$ as follows:

i. All vertices of set V_i have the same color, let say w_i for all $1 \leq i \leq r$, since they are not adjacent to each other.
ii. For every $1 \leq i < i+1 \leq r$, $\varepsilon_{(i,l)}$ is colored by w_i if l is even and it is colored by w_{i+1} if l is odd where $l \leq m$.
iii. Vertex $\varepsilon_{(r,l)}$ is colored by w_r if l is even and it is colored by w_1 if l is odd where $l \leq m$.

Moreover, we have set of colors in Table 1.

Table 1. Set of colors

Sets	Colors
$V_1 \cup P_{(1,m)}$	$\{w_1, w_2\}$
$V_2 \cup P_{(2,m)}$	$\{w_2, w_3\}$
...	...
$V_r \cup P_{(r,m)}$	$\{w_r, w_1\}$

Therefore, the number of colors that are used become minimum and $\chi\big(BT(n,r,m)\big) = r = \chi\big(T(n,r)\big)$ for all $n \geq r \geq 2$ and $m \in \mathbb{N}$. ■

The following proposition provides the diameter of banded-Turán graph.

Proposition 2.9 *For any* $n \geq r \geq 2$ *and* $m \in \mathbb{N}$, *then* $\operatorname{diam}(BT(n,r,m)) = 2m+1$.

Proof. It is easy to know that the maximum distance between any two distinct vertices in $BT(n,r,m)$ is gained by the pair of vertices $(\varepsilon_{(i,m)}, \varepsilon_{(j,m)})$ where $\varepsilon_{(i,m)} \in P_{(i,m)}$, $\varepsilon_{(j,m)} \in P_{(j,m)}$, and $i \neq j$. Moreover, we obtain $d(\varepsilon_{(i,m)}, \varepsilon_{(i,1)}) = m-1 = d(\varepsilon_{(j,m)}, \varepsilon_{(j,1)})$ and since there is a $\varepsilon_{(i,1)} - \varepsilon_{(j,1)}$ path i.e.

$$\varepsilon_{(i,1)}, e_1, \lambda_{(i,1)}, e_2, \lambda_{(j,1)}, e_3, \varepsilon_{(j,1)}$$

such that $d(\varepsilon_{(i,1)}, \varepsilon_{(j,1)}) = 3$. Therefore, $\operatorname{diam}(BT(n,r,m)) = 2(m-1)+3 = 2m+1$. ∎

Lastly, the following proposition provides the metric dimension of banded-Turán graph depending on the number of its classes.

Proposition 2.10 *For* $n \geq r \geq 2$ *and* $m \in \mathbb{N}$, *then* $\beta(BT(n,r,m)) = \begin{cases} n-1, & r=2 \\ \beta(T(n,r)), & r>2. \end{cases}$

Proof. Let $R \subseteq V(BT(n,r,m))$ be a resolving set of $BT(n,r,m)$. For $r=2$, let $V_i = \{\lambda_{(i,j)} : j \in \{1, \dots, n_i\}\}$ where $1 \leq i \leq 2$, $n_1 = \left\lfloor \frac{n}{2} \right\rfloor$, and $n_2 = \left\lceil \frac{n}{2} \right\rceil$.

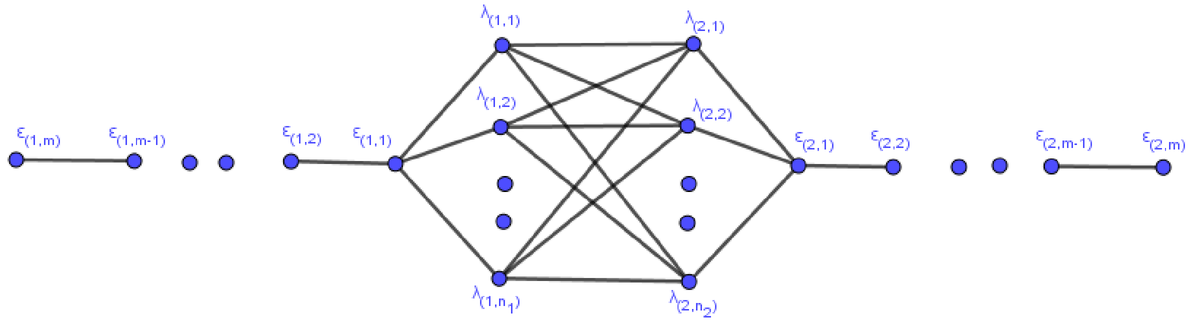

Figure 3. $BT(n,2,m)$

It's easy to see that

$$d(\lambda_{(1,1)}, x) = \cdots = d\left(\lambda_{\left(1,\left\lfloor \frac{n}{2} \right\rfloor\right)}, x\right), \qquad \forall x \in V(BT(n,2,m)) \backslash V_1$$

$$d(\lambda_{(2,1)}, y) = \cdots = d\left(\lambda_{\left(2,\left\lceil \frac{n}{2} \right\rceil\right)}, y\right), \qquad \forall y \in V(BT(n,2,m)) \backslash V_2.$$

By using Lemma 2.4, we obtain $\left\lfloor \frac{n}{2} \right\rfloor - 1$ vertices of V_1 and $\left\lceil \frac{n}{2} \right\rceil - 1$ vertices of V_2 must become landmarks in S so that $|S| \geq n-2$ and say that $\left\{\lambda_{(1,2)}, \dots, \lambda_{\left(1,\left\lfloor \frac{n}{2} \right\rfloor\right)}, \lambda_{(2,2)}, \dots, \lambda_{\left(2,\left\lceil \frac{n}{2} \right\rceil\right)}\right\} \subseteq S$. However, there are two pairs of vertices which have the same coordinate as follows,

$$r(\lambda_{(1,1)}|S) = (2,2, \dots, 2,1,1, \dots, 1) = r(\varepsilon_{(2,1)}|S)$$

$$r(\lambda_{(2,1)}|S) = (1,1, \dots, 1,2,2, \dots, 2) = r(\varepsilon_{(1,1)}|S)$$

and a $\varepsilon_{(1,m)} - \varepsilon_{(2,m)}$ path i.e.

$$\varepsilon_{(1,m)}, e_1, \dots, e_{m-1}, \varepsilon_{(1,1)}, e_m, \lambda_{(1,1)}, e_{m+1}, \lambda_{(2,1)}, e_{m+2}, \varepsilon_{(2,1)}, e_{m+3}, \dots, e_{2m+1}, \varepsilon_{(2,m)}.$$

So, we only need exactly one additional landmark taken from the path, let say $\varepsilon_{(1,1)}$, then we get ordered set

$$S = \left\{\lambda_{(1,2)}, \dots, \lambda_{\left(1,\left\lfloor \frac{n}{2} \right\rfloor\right)}, \lambda_{(2,2)}, \dots, \lambda_{\left(2,\left\lceil \frac{n}{2} \right\rceil\right)}, \varepsilon_{(1,1)}\right\}$$

and all coordinate of vertices are provided in Table 2:

Table 2. Coordinate of vertices in $BT(n,2,m)$

Landmarks	Not landmarks
$r(\lambda_{(1,2)}\|S) = (0,2,2,\dots,2,1,\dots,1)$	$r(\lambda_{(1,1)}\|S) = (2,\dots,2,1,\dots,1)$
$\dots$	$r(\varepsilon_{(1,k)}\|S) = (k,\dots,k,k+1,\dots,k+1,k-1)$, for every $2 \le k \le m$
$r\left(\lambda_{\left(1,\left\lfloor\frac{n}{2}\right\rfloor\right)}\middle\|S\right) = (2,\dots,2,0,1,\dots,1)$	
$r(\lambda_{(2,2)}\|S) = (1,\dots,1,0,2,\dots,2)$	$r(\lambda_{(2,1)}\|S) = (1,\dots,1,2,\dots,2)$
$\dots$	
$r\left(\lambda_{\left(2,\left\lceil\frac{n}{2}\right\rceil\right)}\middle\|S\right) = (1,\dots,1,2,\dots,2,0,2)$	$r(\varepsilon_{(2,k)}\|S) = (k+1,\dots,k+1,k,\dots,k,k+2)$, for every $1 \le k \le m$
$r(\varepsilon_{(1,1)}\|S) = (1,\dots,1,2,\dots,2,0)$	

The set S becomes a basis of $BT(n,2,m)$ and the metric dimension of $BT(n,2,m)$ is $\beta(BT(n,2,m)) = |S| = n-1$. Note that we cannot use an equation $\left\lceil\frac{n}{2}\right\rceil = \left\lfloor\frac{n}{2}\right\rfloor + 1$ for the previous result since the equation does not satisfy for n is divided by 2.

Next for $r > 2$. Since $T(n,r)$ is a part of $BT(n,r,m)$, then let S_1 is a resolving set of $T(n,r)$ for $n \ge r > 2$. Based on Proposition 2.5, if $T(n,r)$ has no class containing exactly one vertex then its basis is the union of each classes removed one vertex i.e. $S_1 = \bigcup_{i=1}^{r}\left(V_i \setminus \{\lambda_{(i,n_i)}\}\right)$ and if $T(n,r)$ has $\delta > 0$ classes containing exactly one vertex then the elements of S_1 are $\delta - 1$ classes containing exactly one vertex and $\bigcup_i\left(V_i \setminus \{\lambda_{(i,n_i)}\}\right)$ for all i where $n_i > 1$. Furthermore, the existence of classes in $T(n,r)$ containing exactly one vertex does not change the following facts:

i. Proposition 2.2.
ii. $r\left(\lambda_{(i_1,n_{i_1})}|S\right) \ne r\left(\lambda_{(i_2,n_{i_2})}|S\right)$ and $r\left(\varepsilon_{(i_1,k)}|S\right) \ne r\left(\varepsilon_{(i_2,k)}|S\right)$ since both are distinct at i_1th and i_2th entry from the coordinate, where $1 \le i_1 < i_2 \le r$ and $1 \le k \le m$.

In other words, every vertex in $BT(n,r,m)$ has a unique coordinate that respect to S_1, so that $S = S_1$ is the basis of $BT(n,r,m)$ for $n \ge r > 2$. Therefore, $\beta(BT(n,r,m)) = \beta(T(n,r))$. ■

CONCLUSION

A banded-Turán graph $BT(n,r,m)$ is a graph formed by a Turán graph and r uniform path graphs in a way that for every vertex in each class of Turán graph is joined by an edge to an initial vertex of a corresponding path graph. We obtain some graph invariants of $BT(n,r,m)$ including independence number, chromatic number, and diameter. Lastly, we obtain that the metric dimension of $BT(n,r,m)$ turns out that it depends on the number of its classes (r). If $r = 2$ then the metric dimension is equal to $n-1$ and if $r > 2$ then the metric dimension is equal to the metric dimension of Turán graph $T(n,r)$.

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