ON THE CONVERGENCE THEOREMS OF THE McSHANE INTEGRAL FOR RIESZ-SPACES-VALUED FUNCTIONS DEFINED ON REAL LINE

Muslim Ansori[1], Yosephus D. Sumanto[2], Novi Rustiana Dewi[3]

[1)]Mathematics Departement, Universitas Lampung
Jln. Soemantri Brodjonegoro No. 1 Bandar Lampung. Email : ansomath@yahoo.com
[2)]Mathematics Departement, Universitas Diponegoro
Jln.Prof.H. Soedarto,SH,Tembalang,Semarang. Email :sumanto_123@yahoo.com
[3)]Mathematics Dept. Universitas Sriwijaya,Palembang

Abstract

This paper is a partial result of our researchs in the main topic "On The McShane Integral for Riesz-Spaces-valued Functions Defined on the space R". We have constructed McShane integral for Riesz-spaces-valued functions defined on the space R by a technique involving double sequences and proved some basic properties which coincides with the McShane Integral for Banach-spaces valued functions defined on real line. Further, we construct some convergence theorems involving uniformly convergence theorems, monotone convergence theorems and Fatou's lemma in the sense of this integral.

Keywords : *Riesz Space, McShane Integral*

1. INTRODUCTION

The recent results of convergence theorems in Integral theory were the Henstock-Kurzweil integral and the McShane integral. Since, the Henstock-Kurzweil integral for Riesz-space-valued functions defined on bounded subintervals of the real line and with respect to operator-valued measures was investigated by Riecan(1989,1992) and Riecan and Brabelova(1996), with respect to (D)- convergence (that is a kind of convergence in which the ε-technique is replaced by a technique involving double sequences , see Riecan and Neubrunn(1997)), with respect to the order convergence, see Boccuto(1998) and in Boccuto and Riecan(2004) with respect to the order convergence but the Henstock-Kurzweil integral for Riesz-space-valued functions was defined on unbounded subintervals of the real line and further, Ansori (2007) have also constructed Henstock-Kurzweil integral for Riesz-space-valued functions defined on Euclidean spaces R^n and

McShane integral for Riesz-spaces-valued functions defined on the real line have been constructed by a technique involving double sequences (Sumanto and Ansori,2007), then the convergence theorems are also be significant to be investigated.

The main goal of this paper is to contruct some convergence theorems in the sense the McShane integral for Riesz-spaces-valued functions defined on the real line.

2. PRELIMINARY

Let N be the set of all strictly positive integers, R the set of the real numbers, R^+ be the set of all strictly positive real numbers. Let $a < a_1 < b_1 < a_2 < b_2 < ... < b$. $[a,b]$ is called interval or cell. $[a_i, b_i], a_i, b_i \in R, a_i < b_i, i = 0,1,2,...,n$. A collection of intervals $[a_i, b_i]$ is called nonoverlapping if their interiors are dijoint. An additive function on a set $[a,b] \subset R$ is a function F defined on the family of all subintervals of $[a,b]$ such that

$$F(B \cup C) = F(B) + F(C)$$

for each pair of nonoverlapping intervals $B, C \subset [a,b]$. A nonnegative additive function ℓ on a set $[a,b]$ is called length on $[a,b]$, defined by $\ell[a,b] = b - a$. Let $P = \{(A_1, x_1), (A_2, x_2), ..., (A_r, x_r)\}$ be a collection of pairs of (A_i, x_i), where $A_1, A_2, ..., A_r$ are nonoverlapping intervals, $\bigcup_{i=1}^{r} A_i = A$ and $A_i \subset O(x_i, \delta(x_i))$, where $O(x_i, \delta(x_i))$ is open ball with center x_i and radius $\delta(x_i)$, $i = 1,2,...,r$. We say that P is $\delta -$fine ***McShane Partition on*** A**.**

The real vector space L (with elements $E_1, E_2, ...$) is called an ordered vector space if L is partially ordered which satisfy $: E_1 \le E_2 \Rightarrow E_1 + h \le E_2 + h$ for every $h \in L$ and $E \ge 0 \Rightarrow kE \ge 0$ for every $k \ge 0$ in R . If, in addition, L is lattice with respect to the partial ordering, then L is called a *Riesz space*. For example, R^n with the familiar coordinate wise addition and scalar multiplication. and by coordinatewise ordering, i.e., for $\bar{x} = (x_1, ..., x_n), \bar{y} = (y_1, ..., y_n)$, we define, $\bar{x} \le \bar{y}$ whenever $x_k \le y_k, k = 1, ..., n$, then R^n is a Riesz space.

Definition 2.1 (Zaanen,1996) **:** *A Riesz space* L *is said to be Dedekind complete if every nonempty subset of* L, *bounded from above, has supremum in* L.

Definition 2.2 (Riecan, 1998) **:** *A bounded double sequence* $(a_{i,j})_{i,j} \in L$ *is called regulator or* (D)*-sequence if, for each* $i \in N, a_{i,j} \downarrow 0$, *that is* $a_{i,j} \geq a_{i,j+1} \forall j \in N$ *and* $\bigwedge_{j \in \mathbb{N}} a_{i,j} = 0$.

Definition 2.3 (Boccuto and Riecan, 2004) **:** *Given a sequence* $(r_n)_n \in L$. *Sequence* $(r_n)_n$ *is said to be* (D)*-convergence to an element* $r \in L$ *if there exist a regulator* $(a_{i,j})_{i,j}$, *satisfying the following condition* : *for every mapping* $\rho : N \to N$, *denoted by* $\rho \in N^N$ *there exists an integer* n_0 *such that* $|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$ *for all* $n \geq n_0$. *In this case, the notation is denoted by* $(D)\lim_n r_n = r$.

Definition 2.4 (Boccuto and Riecan, 2004) : ***A Riesz Space*** L ***is said to be weakly*** $\sigma-$***distributive if for every*** (D) ***- sequence*** $(a_{i,j})$***, then***

$$\bigwedge_{\rho \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\rho(i)} \right) = 0 .$$

Throughout the paper, we shall always assume that L is Dedekind complete weakly $\sigma-$distributive Riesz space.

Here are some recent results in (Sumanto and Ansori,2007):

Definition 3.1 : *A function* $f : [a,b] \subset R \to L$ *is said to be McShane integrable denoted by* $f \in M([a,b], L, \ell)$, *if there exists an element* $E \in L$ *and* (D)*-sequence* $(a_{i,j})_{i,j} \in L$ *such that for every* $\rho \in N^N$ *we can find a function* $\delta : [a,b] \to R^+$ *such that*

$$\left| P\sum f(x)\ell(I) - E \right| = \left| \sum_{k=1}^{r} f(x_k)\ell(I_r) - E \right| \leq \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

for every δ*-fine* M *partition* $P = \{([a,b], x)\} = \{(I_1, x_1), (I_2, x_2), ..., (I_r, x_r)\}$ *on* $[a,b]$.

We note that the McShane integral with respect to ℓ is well- defined, that is there exists at most one element E, satisfying Definition 3.1 and in this case we have $(M)\int_{[a,b]} f dx = E$.

Futhermore we have some fundamental properties of $M(A,L,\ell)$.

Theorem 3.3 : *If* $f_1,f_2 \in M(A,L,\ell)$ *and* $k_1,k_2 \in R$, *then* $k_1f_1 + k_2f_2 \in M(A,L,\ell)$ *and* $(M)\int_{[a,b]}(k_1f_1 + k_2f_2)dx = k_1(M)\int_{[a,b]} f_1dx + k_2(M)\int_{[a,b]} f_2dx$.

Theorem 3.4 : *If* $f,g \in M([a,b],L,\ell)$ *and* $f(x) \le g(x)$ *for every* $x \in A$, *then*

$$(M)\int_{[a,b]} fdx \le (M)\int_{[a,b]} gdx .$$

Definition 3.5 (Elementary Set**):** *A set* $[a,b] \subset R$ *which is union of finite cells is called an elementary set*.

Every elementary set can be segmented into non-overlapping cells. If A_1 and A_2 are elementary sets then $A_1 \cup A_2$ and $\overline{A_1 \setminus A_2}$ are also elementary sets. Integration on elementary set can be constructed through the following theorem.

Teorema 3.6 : *Let* A_1 *and* A_2 *be non-overlapping intervals in* R *and* $A = A_1 \cup A_2$. *If* $f \in M(A_1,L,\ell)$ *and* $f \in M(A_2,L,\ell)$ *, then* $f \in M(A,L,\ell)$ *and*

$$(M)\int_{A=A_1\cup A_2} fdx = (M)\int_{A_1} fdx + (M)\int_{A_2} fdx$$

Corrolary 3.7 : *Given an elementary set* $A \subset R$. *A function* $f : A \to L$ *is said to be McShane integrable on* A *, denoted by* $f \in M(A,L,\ell)$, *if* $f \in M(A_i,L,\ell)$ *for every i, where* $A = \bigcup_{i=1}^{p} A_i$ *and* $\{A_1, A_2, ..., A_r\}$ *is any nonoverlapping intervals of* A. *The McShane integral of function* f *on* A *is*

$$(M)\int_A fdx = \sum_{i=1}^{r}\left[(M)\int_{A_i} fdx\right]$$

The Cauchy criterion was given in the following theorem.

Theorem 3.8 : ***A function*** $f : [a,b] \to L$ ***is McShane integrable if and only if there exists a*** $(D)-$***sequence*** $(a_{i,j})_{i,j}$ ***in*** L ***such that, for every*** $\rho \in N^N$ ***we can find a function***

$\delta:[a,b]\to R^+$ ***and for every*** $\delta-$***fine M- partition*** $P_1=\{([a,b],x)\}$ ***and*** $P_2=\{([a,b],x)\}$ ***on*** $[a,b]$***, we have***

$$\left|P_1\sum f(x)\ell(I)-P_2\sum f(x)\ell(I)\right|\le\bigvee_{i=1}^{\infty}a_{i,\rho(i)}.$$

We provide a result about McShane integrability on subcells.

Theorem 3.9 : *Let* $[a,b]\subset R$*. If* $f\in M(A,L,\ell)$*, then* $f\in M(B,L,\ell)$*, for every interval* $B\subset[a,b]$.

By using Theorem 3.9, we defined primitif function of McShane integrable function f on a cell $[a,b]\subset R$ as follows.

Definition 3.10 : *If* $f\in M([a,b],L,\ell)$ *and* $I([a,b])$ *is a collection of all subcells in* $[a,b]$*, then a function* $F:I([a,b])\to L$ *satisfying*

$$F(J)=(M)\int_J f dx \quad and\ F(\phi)=0$$

for every interval $J\in I(A)$ *is called Primitif of McShane integrable function* f *on* $I([a,b])$.

MAIN RESULTS

Convergence of a sequence of functions is always associated with its limit and function property in the sequence. Let $\{f^{(n)}\}$ be a sequence function defined on $[a,b]\subset R$. Then this sequence is said to be convergent to a function f on $[a,b]$ if $\lim\limits_{n\to\infty}f^{(n)}(x)=f(x)$ for every $x\in[a,b]$. A sequence of functions $\{f^{(n)}\}$ is called increasingly monoton or decreasingly monoton if $f^{(n)}(x)\le f^{(n+1)}(x)$ or $f^{(n)}(x)\ge f^{(n+1)}(x)$ for every $x\in[a,b]$, respectively. Let $f^{(n)}:[a,b]\to L$ for every $n\in N$. Then a sequence function $\{f^{(n)}\}$ is convergent to a function f at $x \in [a,b]$ if and only if there exists (D)-sequence $(a_{i,j})_{i,j} \in L$ such that for every $\rho \in N^N$, we can find $n_0 \in N$, such that

$$\left|f^{(n)}(x) - f(x)\right| < \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

for every $n \geq n_0$. A sequence function $\{f^{(n)}\}$ is convergent to a function f on $[a,b]$ if and only if sequence $\{f^{(n)}(x)\}$ is convergent to a function $f(x)$ for every $x \in [a,b]$. A sequence function $\{f^{(n)}\}$ is uniformly convergent to a function f on A if and only if if there exists (D)-sequence $(b_{i,j})_{i,j} \in L$ such that for every $\rho \in N^N$, we can find $n_0 \in N$, such that

$$\left|f^{(n)}(x) - f(x)\right| < \bigvee_{i=1}^{\infty} b_{i,\rho(i)}$$

for every $n \geq n_0$, $x \in [a,b]$.

Now, we give some convergence theorems for McShane Integrable functions with values in L.

Theorem 4.1.(*Uniformly Convergence Theorem*). *Let* $[a,b] \subset R$ *and* $\{f^{(n)}\} \subset M([a,b],L,\ell)$. *If sequence of functions* $\{f^{(n)}\}$ *is convergent uniformly to a function* f *on* A, *then* $f \in M([a,b],L,\ell)$ *and*

$$(M)\int_{[a,b]} f dx = \lim_{n\to\infty} (M)\int_{[a,b]} f^{(n)} dx$$

Proof: A sequence function $\{f^{(n)}\}$ is uniformly convergent to a function f on $[a,b]$ if and only if if there exists (D)-sequence $(a_{i,j})_{i,j} \in L$ such that for every $\rho \in N^N$, we can find $n_0 \in N$, such that

$$\left|f^{(n)}(x) - f(x)\right| < \tfrac{1}{\alpha(I)} \bigvee_{i=1}^{\infty} a_{i,\rho(i)} \qquad (1.1)$$

for every $n \geq n_0$, $x \in [a,b]$.

A function $f^{(n)} \in M([a,b],L,\ell)$ if there exists (D)-sequence $(b_{i,j})_{i,j} \in L$ such that for every $\rho \in N^N$ we can find a function $\delta : A \to R^+$ such that

$$\left| P\sum_{k=1}^{r} f^{(n)}(x_k)\alpha(I_r) - (M)\int_I f^{(n)} d\alpha \right| \leq \tfrac{1}{\alpha(I)} \mathop{\vee}_{i=1}^{\infty} b_{i,\rho(i)}$$

for every δ-fine McShane partition $P = \{([a,b],x)\} = \{(I_1,x_1),(I_2,x_2),...,(I_r,x_r)\}$ on $[a,b]$.

and if $P_1 = \{(x,[a,b])\}, P_2 = \{(x,[a,b])\}$ are δ_n- fine McShane partition on $[a,b]$, then

$$\left| P_1\sum f^{(n)}(x)\ell(I) - P_2\sum f^{(n)}(x)\ell(I) \right| < \mathop{\vee}_{i=1}^{\infty} c_{i,\rho(i)} \tag{1.3}$$

Based on (1.1) and (1.3), we have

$$\begin{aligned} P_1\sum f(x)\ell(I) - P_2\sum f(x)\ell(I) &\leq \left| P_1\sum \bar{f}(x)\ell(I) - P_1\sum f^{(m)}(x)\ell(I) \right| + \\ &\quad \left| P_1\sum f^{(m)}(x)\ell(I) - P_2\sum f^{(m)}(x)\ell(I) \right| + \\ &\quad \left| P_2\sum f(x)\ell(I) - P_2\sum f^{(m)}(x)\ell(I) \right| \qquad (1.4) \\ &\leq \frac{\mathop{\vee}_{i=1}^{\infty} a_{i,\rho(i)}}{\alpha(I)}\alpha(I) + \mathop{\vee}_{i=1}^{\infty} c_{i,\rho(i)} + \frac{\mathop{\vee}_{i=1}^{\infty} a_{i,\rho(i)}}{\alpha(I)}\alpha(I) \\ &< 2\mathop{\vee}_{i=1}^{\infty} d_{i,\rho(i)} \end{aligned}$$

where $d_{i,\rho(i)} = a_{i,\rho(i)} + c_{i,\rho(i)}$.This shows that $f \in HK([a,b],L,\ell)$. Furthermore, since $f \in M([a,b],L,\ell)$, then there exists (D)-sequence $(b_{i,j})_{i,j} \in L$ such that for every $\rho \in N^N$ we can find a function $\delta' : A \to R^+$ such that

$$\left| \sum_{k=1}^{r} f^{(n)}(\bar{x}_k)\alpha(I_r) - (HK)\int_A f^{(n)} d\alpha \right| \leq \tfrac{1}{\alpha(A)} \mathop{\vee}_{i=1}^{\infty} b_{i,\rho(i)}$$

for every δ-fine McShane partition $P' = \{(A,\bar{x})\}$ on A. By taking $\delta(\bar{x}) = \min\{\delta'(\bar{x}), \delta_n(\bar{x})\}$ for every $\bar{x} \in A$ and if $P = ((A,\bar{x}))$ is arbitrary δ- fine McShane partition on A, then we have

$$\left|(M)\int_A \bar{f}\,d\alpha-(M)\int_A \bar{f}^{(n)}d\alpha\right| \le \left|(M)\int_A \bar{f}\,d\alpha - P\sum \bar{f}\,\alpha(I)\right| + \left|P\sum \bar{f}\,\alpha(I) - P\sum \bar{f}^{(n)}\alpha(I)\right| + \left|P\sum \bar{f}^{(n)}\alpha(I)-(M)\int_A \bar{f}^{(n)}d\alpha\right| \quad (1.6)$$

$$\le \frac{\overset{\infty}{\underset{i=1}{\vee}} b_{i,\rho(i)}}{\alpha(A)}\alpha(A) + \frac{\overset{\infty}{\underset{i=1}{\vee}} a_{i,\rho(i)}}{\alpha(A)}\alpha(A) + \overset{\infty}{\underset{i=1}{\vee}} c_{i,\rho(i)}$$

$$< 2\overset{\infty}{\underset{i=1}{\vee}} e_{i,\rho(i)}$$

where $e_{i,\rho(i)} = a_{i,\rho(i)} + b_{i,\rho(i)} + c_{i,\rho(i)}$. Thus, we have proved that $(M)\int_{[a,b]} f dx = \lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}dx$. □

Theorem 4.2 (*Monoton Convergence Theorem*). *Let* $[a,b]\subset R$ *be a cell and* $\{f^{(n)}\}\subset([a,b],L,\ell)$. *If sequence of monoton functions* $\{f^{(n)}\}$ *is convergent to a function* f *on* $[a,b]$ *and* $\lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}dx$ *exists, then* $f\in M([a,b],L,\ell)$ *and*

$$(M)\int_{[a,b]} f dx = \lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}dx$$

Proof: We just need to prove for sequence $\{f^{(n)}\}$ of increasingly monoton on $[a,b]$. Based on assumption , we can find $E\in L$ so that $\lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}dx = E$. Since $\{f^{(n)}\}$ of increasingly monoton on $[a,b]$, it follows that this sequence, $\left\{(M)\int_{[a,b]} f^{(n)}dx\right\}$ is increasingly monoton and E as its least upper bound. Hence, there exists (D)-sequence $(a_{i,j})_{i,j}\in L$ such that for every $\rho\in N^N$,there exists $n_0\in N$, such that if $n\ge n_0$, we obtain

$$\left|E-(H)\int_{[a,b]} f^{(n)}dx\right| < \overset{\infty}{\underset{i=1}{\vee}} a_{i,\rho(i)} \quad (2.1)$$

Since $\{f^{(n)}\}$ is convergent to a function f on $[a,b]$, then there exists (D)-sequence $(b_{i,j})_{i,j}\in L$ such that for every $\rho\in N^N$ and $x\in[a,b]$ there exists $m_0 = m_0(\varepsilon,x)$ such that if

$n \geq m_0$, we have

$$\left|f^{(n)}(x) - f(x)\right| < \bigvee_{i=1}^{\infty} b_{i,\rho(i)} \tag{2.2}$$

Since $f^{(n)} \in M([a,b],L,\ell)$ for every n, it follows that there exists (D)-sequence $(c_{i,j})_{i,j} \in L$ such that for every $\rho \in N^N$, we can find a positive function $\delta_n : A \to (0,\infty)$ such that for every δ_n-fine McShane partition $P_n = (([a,b],x))$ on $[a,b]$, we have

$$\left|P_n \sum f(x)\ell(I) - F^{(n)}(A)\right| < \frac{\bigvee_{i=1}^{\infty} c_{i,\rho(i)}}{2^{n+2}} \tag{2.3}$$

Taking positive function $\delta : A \to (0,\infty)$ where $\delta(x) = \delta_{m(\varepsilon,\bar{x})}(x)$ for every $x \in [a,b]$ and $m(\varepsilon,x) = \max\{n_0, m_0(\varepsilon,x)\}$. Hence, if $P = (([a,b],x))$ is δ-fine McShane partition on $[a,b]$, then

$$\begin{aligned}
\left|P\sum f(x)\ell(I) - E\right| &= \left|\sum_{i=1}^{k} f(x_i)\ell(I_i) - E\right| \\
&\leq \left|\sum_{i=1}^{k}\left\{f(x_i)\ell(I_i) - f^{(m(\varepsilon,x_i))}(x_i)\ell(I_i)\right\}\right| + \\
&\quad \left|\sum_{i=1}^{k}\left\{f^{(m(\varepsilon,\bar{x}_i))}(x_i)\ell(I_i) - (M)\int_{I_i} f^{(m(\varepsilon,x_i))}dX\right\}\right| \\
&\quad + \left|\sum_{i=1}^{k}(M)\int_{I_i} \bar{f}^{(m(\varepsilon,x_i))}dx - E\right| \\
&< \frac{\bigvee_{i=1}^{\infty} b_{i,\rho(i)}}{\alpha(I)} \sum_{i=1}^{k} \alpha(I_i) + 2\sum_{n=1}^{\infty} \frac{\bigvee_{i=1}^{\infty} c_{i,\rho(i)}}{2^{n+2}} + \bigvee_{i=1}^{\infty} a_{i,\rho(i)} \\
&< 2\bigvee_{i=1}^{\infty} f_{i,\rho(i)}
\end{aligned} \tag{2.4}$$

where, $f_{i,\rho(i)} = \bigvee_{i=1}^{\infty} a_{i,\rho(i)} + \bigvee_{i=1}^{\infty} b_{i,\rho(i)} + \bigvee_{i=1}^{\infty} c_{i,\rho(i)}$.

This shows that $f \in M([a,b],L,\ell)$ and $\lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}d = (M)\int_{[a,b]} fdx = E$. □

Furthermore, Let $\inf\{f^{(n)}\} = \{\inf(f_1^{(n)}), \inf(f_2^{(n)}),\ldots\}$ be infimum of sequence $\{f^{(n)}\}$.

Theorem 4.3. Let $[a,b] \subset R$ be a cell. If $f^{(n)}, g_n \in M([a,b],L,\ell)$ and $f^{(n)} \geq g$ for every n, then $\inf\{f^{(n)}\} \in M([a,b],L,\ell)$.

Proof: Let $h^{(n)}(x)=\min\{f^{(1)}(x),f^{(2)}(x),...,f^{(n)}(x)\}$ be defined for every $x\in[a,b]$. It follows that $h^{(n)}(x)\ge g(x)$ for every $x\in[a,b]$. Thus, we have a decreasingly bounded sequence $\{h^{(n)}\}$ where $\inf\{f^{(n)}\}=h(x)\ge g(x)$ be its bound for every $x\in[a,b]$. Since $f^{(n)}\in M([a,],L,\ell)$ for every n, then $h^{(n)}\in M([a,b],L,\ell)$ and since $g\le h^{(n)}\le f^{(k)}$ for every $n\ge k$, it follows that

$$(M)\int_{[a,b]} gd\alpha \le (M)\int_{[a,b]} h^{(n)}dx \le (M)\int_{[a,b]} f^{(k)}dx \tag{3.1}$$

and $\lim_{n\to\infty}(M)\int_{[a,b]} h^{(n)}dx$ exists, say $E=\lim_{n\to\infty}(M)\int_{[a,b]} h^{(n)}dx$. Thus, $(M)\int_{[a,b]} gdx \le \bar{a} \le (M)\int_{[a,b]} f^{(k)}dx$ for every k. And, from Theorem 2, this shows that $\inf\{f^{(n)}\}\in M([a,b],L,\ell)$. □

Theorem 4.4. (*Fatou's Lemma*). Let $[a,b]\subset R$ and $f^{(n)}\in M([a,b],L,\ell)$ for every n. If $f=\lim_{n\to\infty}\inf_{n\ge k}\{f^{(k)}\}$ and $f=\lim_{n\to\infty}\inf_{n\ge k}\left\{(M)\int_{[a,b]} f^{(n)}dx\right\}<+\infty$, then $f\in M([a,b],L,\ell)$ and

$$(M)\int_{[a,b]} fdx \le \lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}dx$$

Proof: Let $h^{(n)}(x)=\inf_{k\ge n}\{f^{(k)}\}$ be defined for every $x\in[a,b]$. It follows that $\{h^{(n)}\}$ is increasingly sequence on A and $h^{(n)}(x)\le f^{(n)}(x)$ for every $x\in[a,b]$. Thus,

$$\lim_{n\to\infty}(M)\int_{[a,b]} h^{(n)}dx \le \lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}dx \tag{4.1}$$

and $\{h^{(n)}\}$ is convergent to f. Since $\lim_{n\to\infty}(M)\int_{[a,b]} h^{(n)}dx$ exists, then under Monoton convergence Theorem , we have $f\in([a,b],L,\ell)$ and

$$(M)\int_{[a,b]} fdx = \lim_{n\to\infty}(M)\int_{[a,b]} h^{(n)}dx \le \lim_{n\to\infty}\inf_{n\ge k}\left\{(M)\int_{[a,b]} f^{(n)}dx\right\}. \square$$

Corrolary 4.5 . Let $[a,b]\subset R$ be a cell, $f^{(n)},g_n\in M([a,b],L,\ell)$ and $-g\le f^{(n)}\le g$ for every n. If $\lim_{n\to\infty} f^{(n)}=f$ then $f\in([a,b],L,\ell)$ and $(M)\int_{[a,b]} fdx=\lim_{n\to\infty}(M)\int_{[a,b]} f^{(n)}dx$.

CONCLUDING REMARKS

The convergence theorems for the McShane integral for Riesz-space-valued functions defined on real line have been built by generalizing the results in the McShane integral for bounded-sequence-space-valued functions defined on real line and by using by a technique involving double sequences.

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