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The Complexity of Pencil Graph and Line Pencil Graph

Calvin Hanssen, Fransiskus Fran*, Yundari
Departement Mathematics, Universitas Tanjungpura
* Corresponding Author. E-mail: fransiskusfran@math.untan.ac.id

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ABSTRACT

Let $\mathcal{G}$ be a connected undirected graph. Every connected graph $\mathcal{G}$ must contain a spanning tree $\mathcal{T}$, which is a subgraph of $\mathcal{G}$ that is a tree and includes all the vertices of $\mathcal{G}$. The number of spanning trees in $\mathcal{G}$, also called the complexity of the graph $\mathcal{G}$ denoted by $\tau(\mathcal{G})$, is the total number of distinct spanning trees of $\mathcal{G}$. This research aims to formulate the complexity of pencil graph and line pencil graph. In this research, the complexity of pencil graph and line pencil graph are determined using the graph complement approach. The result of the research are the complexity of pencil graph and line pencil graph.

Misalkan $\mathcal{G}$ adalah sebuah graf tak-terarah terhubung. Setiap graf terhubung $\mathcal{G}$ pasti mengandung sebuah pohon merentang $\mathcal{T}$, yaitu sebuah subgraf dari $\mathcal{G}$ yang merupakan sebuah pohon dan memuat semua simpul dari $\mathcal{G}$. Jumlah pohon merentang dalam $\mathcal{G}$, disebut juga kompleksitas graf $\mathcal{G}$, yang dilambangkan dengan $\tau(\mathcal{G})$. Penelitian ini bertujuan untuk memformulasikan kompleksitas dari graf pensil dan graf garis dari graf pensil. Pada penelitian ini, kompleksitas graf pensil dan graf garis dari graf pensil ditentukan dengan menggunakan pendekatan komplemen graf. Hasil dari penelitian ini adalah kompleksitas dari graf pensil dan graf garis dari graf pensil.

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INTRODUCTION

Graph theory is a discipline of discrete mathematics that can be implemented in many different areas. Graph theory has applications in computer science, industry, security, government, and others (Ali et al., 2022; Joedo et al., 2022; Verma et al., 2022). One of the discussions in graph theory that is still a frequently studied topic is the complexity of the graph.

The complexity of a connected undirected specific graph $\mathcal{G}$, denoted by $\tau(\mathcal{G})$. There are numerous techniques for counting the complexity of the graph. Kirchhoff was the first to examine the matter of counting spanning trees of a graph and provided the Matrix-Tree theorem for calculating the complexity of any connected graph (Zhang & Yan, 2020). The Kirchhoff Matrix Tree Theorem is a simple and effective method for finding the complexity in a graph $\mathcal{G}$ using determinant. However, evaluating the appropriate determinant in a sizeable generic graph is time-consuming and computationally inefficient. Therefore, different techniques have been developed to count the number of spanning trees in certain graphs (Daoud, 2019b, 2019a).

Many researchers have developed the concept of complexity graphs by applying it to various types of graphs. Daoud has obtained several results on the number of spanning trees of graphs formed by wheel graph and their asymptotic limits using the recurrence relation and deletion-contraction method (Daoud, 2017). Meanwhile, Daoud and Mohamed have done a related study that calculated the number of spanning trees of some families of cycle-related graphs using the graph complement approach (Daoud & Mohamed, 2017).

The complexity of some families of graphs formed by a triangle using the same approach is discussed by (Daoud, 2019a, 2019b). Zeen El Deen and Aboamer have determined the complexity of some duplicating networks (Zeen El Deen & Aboamer, 2021). In addition, several previous studies focus on examining the complexity of another various graphs , (Afzal et al., 2022; Daoud, 2014; Daoud & Saleh, 2020; Fran et al., 2024; Zeen El Deen, 2023; Zeen El Deen & Aboamer, 2021) This research adopts the method (based on the graph complement approach) to calculate the complexity of the graph under study. This research determined the complexity of some specific graph families (pencil graph Pc_n and line pencil graph $\mathcal{L}(Pc_n)$).

METHODS

This research applies the graph complement approach to determine the complexity of some specific graph families. The complexity of the graph $\mathcal{G}$, denoted by $\tau(\mathcal{G})$, is the number of spanning trees in $\mathcal{G}$. The following explains several definitions and lemmas that define a method for computing the complexity of pencil graph Pc_n and line pencil graph $\mathcal{L}(Pc_n)$.

Definition 1. (Chartrand et al., 2015) A graph $\mathcal{G}$ is a possibly empty set $E(\mathcal{G})$ consisting of two element subsets of $V(\mathcal{G})$ called edges and a specific non-empty set $V(\mathcal{G})$ having elements called vertices (the singular form is vertex). Symbols $|V(\mathcal{G})|$ for the number of vertices and $|E(\mathcal{G})|$ for the number of edges. $|V(\mathcal{G})|$ represents the order of a graph $\mathcal{G}$.

Definition 2. A graph $\mathcal{H}$ is subgraph of a graph $\mathcal{G}$ if $V(\mathcal{H}) \subseteq V(\mathcal{G})$ and $E(\mathcal{H}) \subseteq E(\mathcal{G})$. If $V(\mathcal{H}) \subseteq V(\mathcal{G})$ and $E(\mathcal{H}) \subseteq E(\mathcal{G})$, then a graph $\mathcal{H}$ is subgraph of a graph $\mathcal{G}$.

As an example, below is an illustration of graph $\mathcal{G}_1$ and $\mathcal{G}_2$.

Figure 1. (a) Graph $\mathcal{G}_1$ and (b) Graph $\mathcal{G}_2$

Figure 1 (a) illustrated graph $\mathcal{G}_1$ which has $|V(\mathcal{G}_1)| = 4$ and $|E(\mathcal{G}_1)| = 6$, with a vertex set $V(\mathcal{G}) = \{v_1, v_2, v_3, v_4\}$ and an edge set $E(\mathcal{G}_1) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. **Figure 1 (b)** illustrated graph $\mathcal{G}_2$. Graph $\mathcal{G}_2$ is a subgraph of graph $\mathcal{G}_1$ because all vertices and edges in graph $\mathcal{G}_2$ are contained in graph $\mathcal{G}_1$.

Definition 3. A tree is a connected undirected graph that has no cycles.

The following examples of tree and non-tree are illustrated in **Figure 2**.

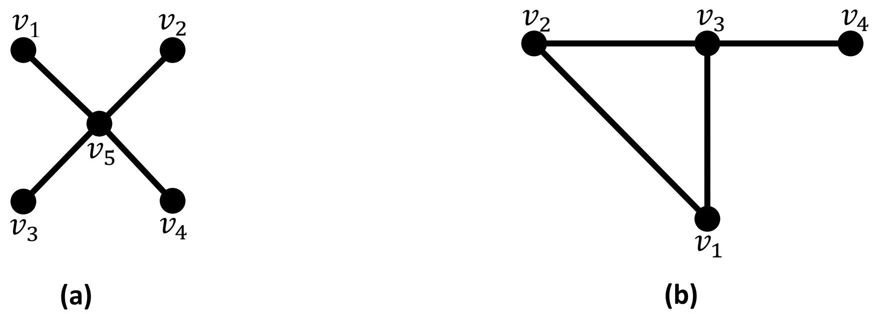

Figure 2. (a) Graph $\mathcal{G}_3$ is tree and (b) Graph $\mathcal{G}_4$ is non-tree

Figure 2 (a) illustrated graph $\mathcal{G}_3$ which is a tree. **Figure 2 (b)** illustrated graph $\mathcal{G}_4$. Graph $\mathcal{G}_4$ is not a tree because it contains a cycle.

Definition 4. If a subgraph $\mathcal{T}$ of connected graph $\mathcal{G}$ is a tree that includes all the vertices of $\mathcal{G}$, then $\mathcal{T}$ is a spanning tree of $\mathcal{G}$.

Based on **Figure 2 (b)**, graph $\mathcal{G}_4$ is a connected graph and its spanning trees are illustrated in **Figure 3**.

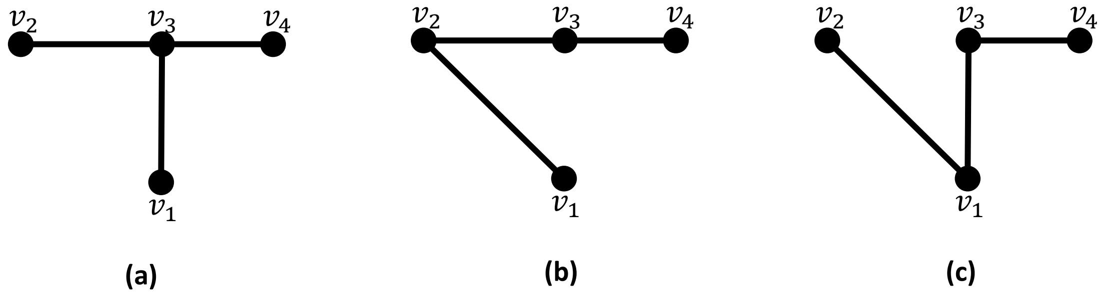

Figure 3. (a) Tree $\mathcal{T}_1$, (b) Tree $\mathcal{T}_2$, and (c) Tree $\mathcal{T}_3$

Definition 5. (Simamora & Salman, 2015) Let n be a positive integer with $n \geq 2$. A pencil graph, denoted by Pc_n, is a graph that has $2n + 2$ vertices, with the vertex set and the edge set as follows.

$$V(Pc_n) = \{u_k, v_k | k \in [0, n]\}$$

$$E(Pc_n) = \{u_k u_{k+1}, v_k v_{k+1} | k \in [1, n-1]\} \cup \{u_k v_k | k \in [0, n]\} \cup \{u_0 u_1, u_0 v_1, u_n v_0, v_n v_0\}$$

As an illustration, the pencil graph $\boldsymbol{Pc_n}$ as shown in **Figure 4**.

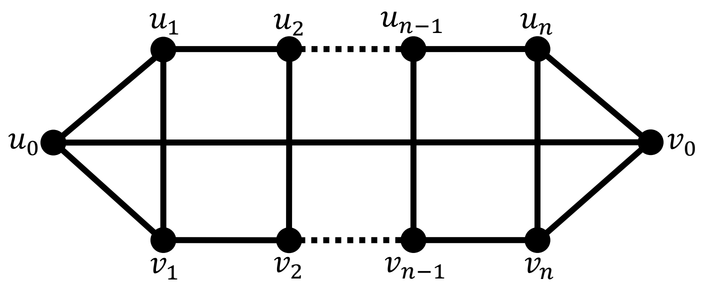

Figure 4. Pencil Graph Pc_n

Definition 6. Let $\mathcal{G}$ be a graph with vertex set $V(\mathcal{G})$ and edge set $E(\mathcal{G})$. The line graph $\mathcal{L}(\mathcal{G})$ is a graph with $V(\mathcal{L}(\mathcal{G})) = E(\mathcal{G})$ and vertices in $\mathcal{L}(\mathcal{G})$ are adjacent if and only if the related edges are incident in $\mathcal{G}$.

The following line pencil graph $\mathcal{L}(Pc_n)$ are illustrated in **Figure 5.**

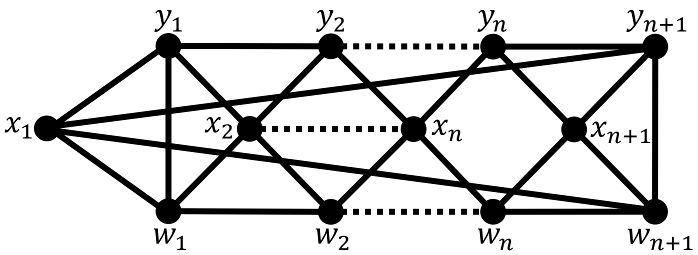

Figure 5. Line Pencil Graph $\mathcal{L}(Pc_n)$

Lemma 1. (Daoud & Saleh, 2020) Let $\mathcal{G}$ be a graph with n vertices and $\bar{\mathcal{G}}$ is the complement of $\mathcal{G}$. Then,

$$\tau(\mathcal{G}) = \frac{1}{n^2}\det\left(nI - D(\bar{\mathcal{G}}) + A(\bar{\mathcal{G}})\right)$$

with $A(\bar{\mathcal{G}})$ are the adjacency matrix of $\bar{\mathcal{G}}$, $D(\bar{\mathcal{G}})$ are degree matrix of $\bar{\mathcal{G}}$ and I is the $n \times n$ identity matrix.

Lemma 2. Let $A_m(\chi)$ be an $m \times m$ matrix, $m \geq 1$, $\chi^2 \neq 1$ such that

$$A_m(\chi) = \begin{bmatrix} 2\chi & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2\chi & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2\chi & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2\chi \end{bmatrix}$$

then,

$$\det\left(A_m(\chi)\right) = \frac{1}{2\sqrt{\chi^2 - 1}}\left[\left(\chi + \sqrt{\chi^2 - 1}\right)^{m+1} - \left(\chi - \sqrt{\chi^2 - 1}\right)^{m+1}\right].$$

Lemma 3. (Zeen El Deen & Aboamer, 2021) Let $B_m(\chi)$ be an $m \times m$ circulant matrix, $\chi \geq 2$ such that

$$B_m(\chi) = \begin{bmatrix} \chi & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & \chi \end{bmatrix}$$

then, $\det\left(B_m(\chi)\right) = (\chi + m - 1)(\chi - 1)^{m-1}$.

Lemma 4. (Deen et al., 2024) Let $\mathcal{A}, \mathcal{B} \in \mathbb{F}^{n \times n}$ with $\mathcal{AB} = \mathcal{BA}$ and $\mathcal{C} \in \mathbb{F}^{2n \times 2n}$ such that

$$\mathcal{C} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{bmatrix}$$

then, $\det(\mathcal{C}) = \det(\mathcal{A} - \mathcal{B})\det(\mathcal{A} + \mathcal{B})$.

Lemma 5. (Javaid et al., 2021) Let $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be matrices of dimension $n \times n$, $n \times m$, $m \times n$, and $m \times m$, respectively. If K and N are invertible, then

$$\det \begin{bmatrix} \mathcal{K} & \mathcal{L} \\ \mathcal{M} & \mathcal{N} \end{bmatrix} = \det \mathcal{K} \det(\mathcal{N} - \mathcal{M}\mathcal{K}^{-1}\mathcal{L}) = \det \mathcal{N} \det(\mathcal{K} - L\mathcal{N}^{-1}\mathcal{M})$$

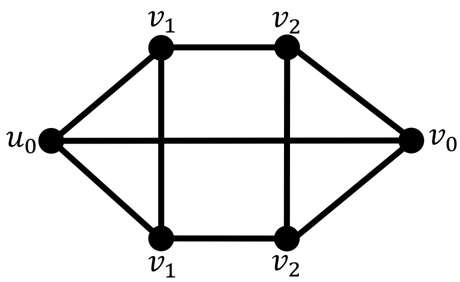

Figure 6. Pencil Graph Pc_2

Example 1. Let Pc_2 be a graph with 6 vertices as illustrated in **Figure 6**. Then, the number of spanning trees of Pc_2 is obtained by the following steps.

First, it can be obtained the $A(\overline{Pc_2})$ and $D(\overline{Pc_2})$ from **Figure 6**.

$$A(\overline{Pc_2}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } D(\overline{Pc_2}) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

From $A(\overline{Pc_2})$ and $D(\overline{Pc_2})$, matrix $6I - D(\overline{Pc_2}) + A(\overline{Pc_2})$ can be formed.

$$6I - D(\overline{Pc_2}) + A(\overline{Pc_2}) = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 & 1 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \\ 1 & 0 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 1 & 4 & 0 \\ 1 & 0 & 1 & 0 & 0 & 4 \end{bmatrix}$$

Then, applying Lemma 1 yields

$$\begin{aligned} \tau(Pc_2) &= \frac{1}{6^2} \det[6I - D(\overline{Pc_2}) + A(\overline{Pc_2})] \\ &= \frac{1}{36} \det \begin{bmatrix} 4 & 0 & 0 & 1 & 0 & 1 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \\ 1 & 0 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 1 & 4 & 0 \\ 1 & 0 & 1 & 0 & 0 & 4 \end{bmatrix} = \frac{1}{36} \det \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \end{aligned}$$

Applying Lemma 5 yields

$$\begin{aligned}\tau(Pc_2) &= \frac{1}{36}\det(A)\det(C - B^T A^{-1} B) \\ &= \frac{1}{36}\det\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\det\left(\begin{bmatrix} \frac{15}{4} & 1 & -\frac{1}{4} \\ 1 & \frac{56}{15} & \frac{1}{15} \\ -\frac{1}{4} & -\frac{1}{15} & \frac{31}{60} \end{bmatrix}\right) = \frac{1}{36}(60)(45) = \frac{1}{36}(2700) \\ &= 75\end{aligned}$$

Thus, the complexity of Pc_2 is 75.

RESEARCH AND DISCUSSION

This research produces two theorems on the complexity of pencil graph Pc_n and line pencil graph $\mathcal{L}(Pc_n)$, as follows.

3.1 Complexity of Pencil Graph

Complexity of pencil graph is referred to as the number of spanning trees of pencil graph. A subgraph $\mathcal{T}$ is a spanning tree of pencil graph if $\mathcal{T}$ is a tree that includes all the vertices of pencil graph. The following is a proof of the complexity theorem on the pencil graph.

Theorem 1. Given a pencil graph Pc_n, then the complexity of pencil graph Pc_n with $n \geq 2$ is

$$\tau(Pc_n) = \frac{n+3}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]$$

Proof. Order of the pencil graph Pc_n is $|V(Pc_n)| = 2n + 2$ and number of edges are $|E(Pc_n)| = 3n + 3$, see Figure 4. Applying Lemma 1 yields

$$\begin{aligned}\tau(Pc_n) &= \frac{1}{(2n+2)^2}\det[(2n+2)I - D(\overline{Pc_n}) + A(\overline{Pc_n})] \\ &= \frac{1}{(2n+2)^2}\det\begin{bmatrix} 4 & 0 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & 4 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \vdots & 0 & 4 & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & \vdots & \ddots & \ddots & \ddots & 4 & 0 & \vdots & \ddots & \ddots & \ddots & 0 & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 4 & 1 & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 4 & 0 & 1 & \cdots & \cdots & 1 \\ 1 & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & 0 & 4 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & \vdots & \ddots & \ddots & \ddots & 0 & 1 & \vdots & \ddots & \ddots & \ddots & 4 & 0 \\ 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 4 \end{bmatrix} \\ &= \frac{1}{(2n+2)^2}\det\begin{bmatrix} A_{2\times 2} & B_{2\times 2n} \\ B^T_{2n\times 2} & C_{2n\times 2n} \end{bmatrix}\end{aligned}$$

Applying Lemma 5 yields

$$\begin{aligned}\tau(Pc_n) &= \frac{1}{(2n+2)^2}\det(A)\det(C - B^T A^{-1} B)\\ &= \frac{1}{(2n+2)^2}\det\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\det\left(\frac{1}{4}\begin{bmatrix}P & Q\\ Q & P\end{bmatrix}_{2n\times 2n}\right)\\ &= \frac{4^{2-2n}}{(2n+2)^2}\det\left(\begin{bmatrix}P & Q\\ Q & P\end{bmatrix}_{2n\times 2n}\right)\end{aligned}$$

Where,

$$P = \begin{bmatrix}15 & -1 & 3 & \cdots & \cdots & 3 & 4\\ -1 & 14 & -2 & 2 & \cdots & 2 & 3\\ 3 & -2 & \ddots & \ddots & \ddots & \vdots & \vdots\\ \vdots & 2 & \ddots & \ddots & \ddots & 2 & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & -2 & 3\\ 3 & 2 & \cdots & 2 & -2 & 14 & -1\\ 4 & 3 & \cdots & \cdots & 3 & -1 & 15\end{bmatrix}_{n\times n} \text{ and } Q = \begin{bmatrix}-1 & 3 & \cdots & \cdots & 3 & 4\\ 3 & -2 & 2 & \cdots & 2 & 3\\ \vdots & 2 & \ddots & \ddots & \vdots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 2 & \vdots\\ 3 & 2 & \cdots & 2 & -2 & 3\\ 4 & 3 & \cdots & \cdots & 3 & -1\end{bmatrix}_{n\times n}$$

By applying Lemma 4, obtained results

$$\begin{aligned}\tau(Pc_n) &= \frac{4^{2-2n}}{(2n+2)^2}\det(P-Q)\det(P+Q)\\ &= \frac{4^{2-2n}}{(2n+2)^2}\times\det\begin{bmatrix}16 & -4 & 0 & \cdots & 0\\ -4 & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -4\\ 0 & \cdots & 0 & -4 & 16\end{bmatrix}_{n\times n}\times\det\begin{bmatrix}14 & 2 & 6 & \cdots\cdots & 6 & 8\\ 2 & 12 & 0 & 4\cdots & 4 & 6\\ 6 & 0 & \ddots & \ddots\ \ddots & \vdots & \vdots\\ \vdots & 4 & \ddots & \ddots\ \ddots & 4 & \vdots\\ \vdots & \vdots & \ddots & \ddots\ \ddots & 0 & 6\\ 6 & 4 & \cdots 4 & 0 & 12 & 2\\ 8 & 6 & \cdots\cdots & 6 & 2 & 14\end{bmatrix}_{n\times n}\\ &= \frac{4^{2-2n}}{(2n+2)^2}\times\left(4^n\det\begin{bmatrix}4 & -1 & 0 & \cdots & 0\\ -1 & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -1\\ 0 & \cdots & 0 & -1 & 4\end{bmatrix}_{n\times n}\right)\times\det\begin{bmatrix}14 & 2 & 6 & \cdots\cdots & 6 & 8\\ 2 & 12 & 0 & 4\cdots & 4 & 6\\ 6 & 0 & \ddots & \ddots\ \ddots & \vdots & \vdots\\ \vdots & 4 & \ddots & \ddots\ \ddots & 4 & \vdots\\ \vdots & \vdots & \ddots & \ddots\ \ddots & 0 & 6\\ 6 & 4 & \cdots 4 & 0 & 12 & 2\\ 8 & 6 & \cdots\cdots & 6 & 2 & 14\end{bmatrix}_{n\times n}\end{aligned}$$

By applying Lemma 2 and simple induction using determinant properties, obtained results

$$\begin{aligned}\tau(Pc_n) &= \frac{4^{2-2n}}{(2n+2)^2}\times\left[\frac{4^n}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right)\right]\times\left[2^{2n-4}(2n+2)^2(n+3)\right]\\ &= \frac{n+3}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right].\ \blacksquare\end{aligned}$$

In Example 1, it is obtained that the number of spanning tree of Pc_2 is 75. By applying Theorem 1, the number of spanning trees of Pc_2 is as follows.

$$\begin{aligned}\tau(Pc_2) &= \frac{2+3}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{2+1} - \left(2-\sqrt{3}\right)^{2+1}\right] = \frac{5}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^3 - \left(2-\sqrt{3}\right)^3\right]\\ &= \frac{5}{2\sqrt{3}}\left[30\sqrt{3}\right] = \frac{150}{2}\\ &= 75.\end{aligned}$$

3.2 Complexity of Line Pencil Graph

Complexity of line pencil graph is referred to as the number of spanning trees of line pencil graph. A subgraph $\mathcal{T}$ is a spanning tree of line pencil graph if $\mathcal{T}$ is a tree that includes all the vertices of line pencil graph. The following is a proof of the complexity theorem on the line pencil graph.

Theorem 2. Given a line pencil graph $\mathcal{L}(Pc_n)$, then the complexity of pencil graph $\mathcal{L}(Pc_n)$ with $n \geq 2$ is

$$\tau\big(\mathcal{L}(Pc_n)\big) = 2^{n+1}3^{n-1}(n+3)\sqrt{3}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]$$

Proof. Order of the pencil graph Pc_n is $\left|V\big(\mathcal{L}(Pc_n)\big)\right| = 3n+3$ and number of edges are $\left|E\big(\mathcal{L}(Pc_n)\big)\right| = 6n+6$, see Figure 5. Applying Lemma 1 yields

$$\begin{aligned}
\tau\big(\mathcal{L}(Pc_n)\big) &= \frac{1}{(3n+3)^2}\det\left[(3n+3)I - D\left(\overline{\mathcal{L}(Pc_n)}\right) + A\left(\overline{\mathcal{L}(Pc_n)}\right)\right] \\
&= \frac{1}{(3n+3)^2}\det\begin{bmatrix}
5 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 0 \\
1 & 5 & \ddots & \ddots & \vdots & 0 & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 5 & 1 & \vdots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 5 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 0 & 0 \\
0 & 0 & 1 & \cdots & 1 & 5 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \ddots & \vdots & 0 & 5 & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 0 & 0 & \vdots & \ddots & \ddots & 5 & 0 & \vdots & \ddots & \ddots & 0 & 1 \\
0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 5 & 1 & \cdots & \cdots & 1 & 0 \\
0 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 5 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \vdots & 0 & 5 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \vdots & \ddots & \ddots & 0 & 1 & \vdots & \ddots & \ddots & 5 & 0 \\
0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 5
\end{bmatrix} \\
&= \frac{1}{(3n+3)^2}\det\begin{bmatrix} A_{(n+1)\times(n+1)} & B_{(n+1)\times(2n+2)} \\ B^T_{(2n\times 2)\times(n+1)} & C_{(2n+2)\times(2n+2)} \end{bmatrix}
\end{aligned}$$

Applying Lemma 3 and Lemma 5 yields

$$\begin{aligned}
\tau\big(\mathcal{L}(Pc_n)\big) &= \frac{1}{(3n+3)^2}\det(A)\det(C - B^T A^{-1} B) \\
&= \frac{1}{(3n+3)^2} \times \det\begin{bmatrix}
5 & 1 & \cdots & \cdots & 1 \\
1 & 5 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 5
\end{bmatrix}_{(n+1)\times(n+1)} \times \det\left(\frac{1}{4n+20}\begin{bmatrix} P & Q \\ Q & P \end{bmatrix}_{(2n+2)\times(2n+2)}\right) \\
&= \frac{4^n(n+5)}{(3n+3)^2(4n+20)^{2n+2}}\det\left(\begin{bmatrix} P & Q \\ Q & P \end{bmatrix}_{(2n+2)\times(2n+2)}\right)
\end{aligned}$$

Where,

$$P = \begin{bmatrix}
14n+106 & 11-5n & 36 & \cdots & & \cdots & 36 & 31-n \\
11-5n & 14n+106 & \ddots & 36 & & \cdots & 36 & 36 \\
36 & 11-5n & \ddots & \ddots & & \ddots & \vdots & \vdots \\
\vdots & 36 & \ddots & \ddots & & \ddots & 36 & \vdots \\
\vdots & \vdots & \ddots & \ddots & & \ddots & 11-5n & 36 \\
36 & 36 & \cdots & 36 & 11-5n & & 14n+106 & 11-5n \\
31-n & 36 & \cdots & \cdots & 36 & & 11-5n & 14n+106
\end{bmatrix}_{(n+1)\times(n+1)}$$

and

$$Q = \begin{bmatrix}
6-6n & 31-n & 36 & \cdots & \cdots & 36 & 31-n \\
31-n & 26-2n & 31-n & 36 & \cdots & 36 & 36 \\
36 & 31-n & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & 36 & \ddots & \ddots & \ddots & 36 & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 31-n & 36 \\
36 & 36 & \cdots & 36 & 31-n & 26-2n & 31-n \\
31-n & 36 & \cdots & \cdots & 36 & 31-n & 6-6n
\end{bmatrix}_{(n+1)\times(n+1)}$$

By applying Lemma 4, obtained results

$$\tau(\mathcal{L}(Pc_n)) = \frac{4^n(n+5)}{(3n+3)^2(4n+20)^{2n+2}}\det(P-Q)\det(P+Q)$$

$$= \frac{4^n(n+5)}{(3n+3)^2(4n+20)^{2n+2}} \times \det\begin{bmatrix} 20n+100 & -4n-20 & 0 & \cdots & 0 \\ -4n-20 & 16n+80 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 16n+80 & -4n-20 \\ 0 & \cdots & 0 & -4n-20 & 20n+100 \end{bmatrix}$$

$$\times \det\begin{bmatrix} 8n+112 & 42-6n & 72 & \cdots & \cdots & 72 & 62-2n \\ 42-6n & 12n+132 & \ddots & 72 & \cdots & 72 & 72 \\ 72 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & 72 & \ddots & \ddots & \ddots & 72 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 72 \\ 72 & 72 & \cdots & 72 & \ddots & 12n+132 & 42-6n \\ 62-2n & 72 & \cdots & \cdots & 72 & 42-6n & 8n+112 \end{bmatrix}$$

By applying Lemma 2 and simple induction using determinant properties, obtained results

$$\tau(\mathcal{L}(Pc_n)) = \frac{4^n(n+5)}{(3n+3)^2(4n+20)^{2n+2}} \times (4n+20)^{n+1}\sqrt{3}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]$$
$$\times 2^{n+3}3^{n-1}(3n+3)^2(n+3)(n+5)^n$$
$$= 2^{n+1}3^{n-1}(n+3)\sqrt{3}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]. \blacksquare$$

Example 2. Let $\mathcal{L}(Pc_2)$ be a graph with 9 vertices as illustrated in **Figure 7**. Then, by applying Theorem 2, the number of spanning trees of $\mathcal{L}(Pc_2)$ is as follows.

$$\tau(\mathcal{L}(Pc_2)) = 2^{2+1}3^{2-1}(2+3)\sqrt{3}\left[\left(2+\sqrt{3}\right)^{2+1} - \left(2-\sqrt{3}\right)^{2+1}\right]$$
$$= 2^3 3(5)\sqrt{3}\left[\left(2+\sqrt{3}\right)^3 - \left(2-\sqrt{3}\right)^3\right] = 120\sqrt{3}[30\sqrt{3}]$$
$$= 10800.$$

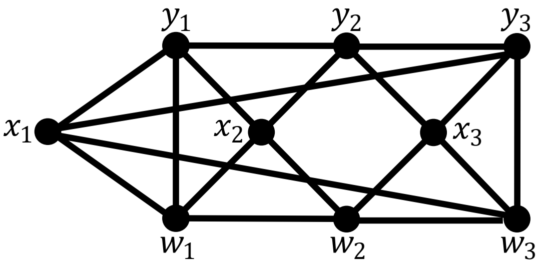

Figure 7. Line Pencil Graph $\mathcal{L}(Pc_2)$

CONCLUSIONS

This research has determined the exact value of complexity of pencil graph and line pencil graph using graph complement approach. The results indicate that this reserach produces two theorems of complexity of the graphs. This research has discovered that the complexity of pencil graph Pc_n is $\tau(Pc_n) = \frac{n+3}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]$, for $n \geq 2$. The complexity of line pencil graph $\mathcal{L}(Pc_n)$ is $\tau(\mathcal{L}(Pc_n)) = 2^{n+1}3^{n-1}(n+3)\sqrt{3}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]$, for $n \geq 2$.

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