CONSTRUCTION A CORING FROM TENSOR PRODUCT OF BIALGEBRA

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Abstract

In this Paper introduced a coring from tensor product of bialgebra. An algebra with compatible coalgebra structure are known as bialgebra. For any bialgebra B we can obtained tensor product between B and itself. Defined a right and left B-action on the tensor product of bialgebra B such that we have tensor product of B and itself is a bimodule over B. In this note we expect that the tensor product B and itself becomes a B-coring with comultiplication and counit.
Keywords : action, algebra, coalgebra, coring.

Abstrak

Pada tulisan ini dikenalkan koring dari hasil kali tensor sebuah bialjabar. Aljabar yang sekaligus juga koajabar disebut bialjabar. Untuk setiap bialjabar B dapat dibentuk hasil kali tensor dari B dengan dirinya sendiri. Didefinisikan B-aksi kiri dan kanan pada hasil kali tensor B tersebut sedemikian hingga hasil kali tensor B dengan dirinya sendiri merupakan modul atas B. Lebih lanjut hasil kali tensor B akan membentuk struktur B-koring yang dilengkapi dengan komultiplikasi dan kounit.
Kata Kunci : aksi, aljabar, koaljabar, koring
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1. INTRODUCTION

Let R is commutative ring. In this paper (B,μ,Δ) will denoted R-module B which is assosiative R-algebra with multiplication μ and unit 1 as well as coassosiative $R-$coalgebra with comultiplication Δ and counit ε. As a R-module, we have tensor product B and itself i.e. $B\otimes_R B$ (Hungerford, 1978). For a left and right action on $B\otimes_R B$ over assosiative (with unit 1) algebra B, $B\otimes_R B$ can be considered as (B,B)-bimodule (left unital). We expect that by define a comultiplication and counit for $B\otimes_R B$, $B\otimes_R B$ becomes a B-coring.

Before continue to the main result, on section 2 we will be given basic concepts about tensor product (Hungerford, 1978). On section 3 we will be given review about algebra, coalgebra and bialgebra (Brzeziński, T., Wisbauer, R., 2003]). Coring is generalized from coalgebra over any ring (associative algebra, not need commutative). Section 4 is the main result in this paper. Section 4 contained an explanation about coring from a tensor product of bialgebra. The author assume that readers have been understanding about module theory. This paper has been presented on "Seminar Nasional Aljabar 2011" Padjajaran University April 30, 2011 in Bandung.

2. TENSOR PRODUCT

In this section we will review about tensor product. It is important to know about bilinear and balanced map before we study tensor product from modules.

Definition 2.1

Let right R-modue M, left R-moduel N and Abelian Group $(G,+)$. A map $\beta : M \times N \to G$ is called (R,R)-bilinear and balanced function provided for any $m_1, m_2 \in M$, $n_1, n_2 \in N$ and $r \in R$

(i). $\beta(m_1+m_2,n_1)=\beta(m_1,n_1)+\beta(m_2,n_1)$

(ii). $\beta(m_1,n_1+n_2)=\beta(m_1,n_1)+\beta(m_1,n_2)$

(iii). $\beta(m_1 r,n_1)=\beta(m_1,rn_1)$ (Balance properties)

The next definition is about tensor product for M and N.

Definition 2.2

Let right R-module M, left R-module N and Abelian Group $(G,+)$. Abelian group $M \otimes_R N$ with bilinear and balanced function τ is called tensor product from M *and* N provided for every bilinear and balanced map $\beta : M \times N \to G$ there is a unique map $\overline{\beta} : M \otimes_R N \to G$ such that diagram above is cummutative diagram, i. e $\beta = \overline{\beta} \circ \tau$.

$$\begin{tikzcd} M \times N \arrow[rr, "\beta"] \arrow[d, "\tau"'] & & G \\ M \otimes_R N \arrow[urr, dashed, "\exists\overline{\beta}"'] & & \end{tikzcd}$$

Figure 1. *Tensor Product Diagram*

Notation for element of $M \otimes_R N$ is $\left\{ \sum_{i \in I} m_i \otimes n_i \middle| m_i \in M \ dan \ n_i \in N \right\}$.

Theorem 2.3

Let right R-module M and left R-module N. Tensor product from M and N is unique.

Lemma 2.4

If $M \otimes_R N$ is tensor product from right R-module M and left R-module N, then for any $m_1, m_2 \in M$, $n_1, n_2 \in N$ and $r \in R$

(i). $(m_1 + m_2) \otimes n_1 = m_1 \otimes n_1 + m_2 \otimes n_1$

(ii). $m_1 \otimes (n_1 + n_2) = m_1 \otimes n_1 + m_1 \otimes n_2$

(iii). $mr \otimes n = m \otimes rn$

(iv). $m \otimes 0 = 0 \otimes n = 0$

Moreover based on tensor product definition, the isomorphism of tensor product will be given on the theorem above.

Theorem 2.8

For every right R-module A and left R-module B, then $A \otimes_R R \cong A$ dan $R \otimes_R B \cong B$.

Theorem 2.9

For every right R-module A, (R,S)-bimodule B and left S-module C, then $(A \otimes_R B) \otimes_S C \cong A \otimes_R (B \otimes_S C)$.

Theorem 2.10

For every right R-module collection $\{A_\lambda\}_{\lambda \in \Lambda}$ and left R-module B, $\left(\sum_{\lambda \in \Lambda} A_\lambda\right) \otimes_R B = \sum_{\lambda \in \Lambda} (A_\lambda \otimes_R B)$.

Definiton 2.11

For any right R-module A and left R-module B, the twist map is defined as a map

$tw : A \otimes_R B \to B \otimes_R A, a \otimes b \mapsto b \otimes a$.

The twist map on Definition 2.11 denoted by tw.

3. ALGEBRA, COALGEBRA AND BIALGEBRA

This section must well known before we study topics in this paper. There are many author write about algebra and coalgebra so it is simple to find definition and some properties about algebra and coalgebra.

Definition 3.1

Let R is a commutative ring. An R-algebra A is a R-module A (with 1) such that it must have multiplication $\mu : A \otimes_R A \to A, a \otimes b \mapsto ab$.

Definition 3.2

(i). An R-algebra A with multiplication μ is called associative algebra provided $\mu \circ (\mu \otimes I_A) = \mu \circ (I_A \otimes \mu)$.

(ii). An R-algebra A with multiplication μ is called algebra with unit if there is homomorphism $\iota : R \to A$ such that $\mu \circ (\iota \otimes I_A) = A = \mu \circ (I_A \otimes \iota)$.

Definiton 3.3

Let A and B be R-algebra. A map $f : A \to B$ is called an R-algebra homomorphism if

(i). f is a ring homomorphism and $f(1_A) = 1_B$.

(ii). f is a module homomorphism.

In this paper an associative R-algebra A with unit ι is denoted by (A, μ, ι). Reasearch about algebra and coalgebra started in 1960 by Sweedler. Colagebra is dual from algebra, it can obtained by reverse the algebra multiplication arrow. The following definition and properties are explain the coalgebra structure.

Definition 3.4.

Let commutative ring R and R-module C. R-module C is called $R-$coalgebra provided it has a R-linear map $\Delta : C \to C \otimes_R C,\ c \mapsto \sum c_{\underline{1}} \otimes c_{\underline{2}}$ and R-linear map $\varepsilon : C \to R$ such that the following diagram is commute.

$$\begin{tikzcd}
 & C \arrow[dl, dashed, "\sqcup"'] \arrow[d, "\Delta"] \arrow[dr, dashed, "\sqcup"] & \\
R \otimes_R C & C \otimes_R C \arrow[l, "\varepsilon \otimes I_C"] \arrow[r, "I_C \otimes \varepsilon"'] & C \otimes_R R
\end{tikzcd}$$

Figure 2. Counit Diagram

A R-linear map Δ on Definition 3.4 is called comultiplication and $\varepsilon : C \to R$ is called counit map. For any $c \in C$, $\sum c_{\underline{1}} \varepsilon(c_{\underline{2}}) = c = \sum \varepsilon(c_{\underline{1}}) c_{\underline{2}}$.

Definition 3.5

Comultiplication $\Delta : C \to C \otimes_R C$ is called coassosiative provided the following diagram is commute.

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d] & C \arrow[d, "\Delta \otimes I_C"] \\
C \otimes_R C \arrow[r, "I_C \otimes \Delta"] & C \otimes_R C \otimes_R C
\end{tikzcd}$$

Figure 3. Coassosiative Diagram

From Definition 3.5 R-coalgebra C becomes a coassositaive coalgebra provided for any $c \in C$, $\sum c_{\underline{1}} \otimes c_{\underline{21}} \otimes c_{\underline{22}} = \sum c_{\underline{11}} \otimes c_{\underline{12}} \otimes c_{\underline{2}}$.

Throughout in this paper R-coalgebra

C over comultilication Δ and counit ε denoted by triple (C,Δ,ε).

Definition 3.6

Coring is R-coaljabar over any ring (not need commutative).

Definition 3.7

Let (C,Δ,ε), $(C',\Delta',\varepsilon')$ and R-module homomorphism $f:C\to C'$. Map $f:C\to C'$ is called coalgebra morphism provided $\Delta'\circ f=(f\otimes f)\circ\Delta$.

Based on Definition 3.4 and 3.5, any R-algebra A can be considered as a coassosiative coalgebra, because we can defined comultiplication $\Delta: A\otimes A\to A, a\mapsto a\otimes a$. If there is a set which is an assosiative algebra as well as coassosiative coalgebra, then it can be a bialgebra.

Definition 3.8

An R-module B that is an algebra (B,μ,ι) and a coalgebra (B,Δ,ε) is called a bialgebra if Δ and ε are algebra morphism or, equivantely, μ and ι are coalgebra morphism.

Definition 3.8, For Δ to be an algebra morphism it means that $\Delta\circ\iota=(\iota\otimes\iota)\circ I_R$ and $\Delta\circ\mu=(\mu\otimes\mu)(I_B\otimes tw\otimes I_B)\circ(\Delta\otimes\Delta)$.

Similarly, ε is an algebra morphism if and only if $\varepsilon\circ\iota=I_R$ and $\varepsilon\circ\mu=I_R\circ(\varepsilon\otimes\varepsilon)$.

4. A CORING FROM A BIALGEBRA

Throughout this section (B,μ,Δ) will denoted an associative R-algebra B with multiplication μ and unit 1 as well as a coassociative coalgebra with comultiplication Δ and counit ε, such that $(\forall a,b\in B)\Delta(ab)=\Delta(a)\Delta(b)$. We always have canonical multiplication with unit $1\otimes 1$ on $B\otimes_R B$ and we will defined comultiplication with counit on $B\otimes_R B$ over associative R-algebra B. For this aim, $B\otimes_R B$ must be a (B,B)-bimodule.

Lemma 4.1

Given R-bialgebra (B,μ,Δ). Tensor product $B\otimes_R B$ is a (B,B)-bimodule with right and (unital) left B-actions with unit 1, i.e

$$ {}_B\alpha : B \times B \otimes_R B \to B \otimes_R B, $$
$$ (a, b \otimes c) \mapsto ab \otimes c $$

$$ \alpha_B : B \otimes_R B \times B \to B \otimes_R B, $$
$$ (a \otimes b, c) \mapsto a \otimes b\Delta(c) = \sum ac_{\underline{1}} \otimes bc_{\underline{2}}. $$

Proof :

From tensor product properties ${}_B\alpha$ and α_B are well defined and its easy to proof the modules axioms in $B \otimes_R B$. For any $a \otimes b \in B \otimes_R B$

and $1 \in B$, $1.(a \otimes b) = 1a \otimes b = a \otimes b.$

Theorem 4.2

For R-bialgebra (B, μ, Δ) define a maps

$$ \underline{\Delta} : B \otimes_R B \to (B \otimes_R B) \otimes_B (B \otimes_R B) \square (B \otimes_R B).1 \otimes_R B, $$
$$ a \otimes b \mapsto \sum (a \otimes b_{\underline{1}}) \otimes_B (1 \otimes b_{\underline{2}}) \mapsto \sum a1_{\underline{1}} \otimes b_{\underline{1}}1_{\underline{2}} \otimes b_{\underline{2}}, $$
$$ \underline{\varepsilon} : B \otimes_R B \to (B \otimes_R B).1 \xrightarrow{I \otimes \varepsilon} B $$
$$ a \otimes b \mapsto (a \otimes b).1 \mapsto (a \otimes b)\Delta(1) \mapsto \sum a1_{\underline{1}} \otimes b1_{\underline{2}} \mapsto \sum a1_{\underline{1}}\varepsilon(b1_{\underline{2}}) $$

Then maps $\underline{\Delta}$ is coassosiative weak comultiplication on $B \otimes_R B$ and the $\underline{\varepsilon}$ is the left B-module morphism with $(a \otimes b).1 = (I_{B \otimes_R B} \otimes \underline{\varepsilon})\underline{\Delta}(a \otimes b)$, for all $a, b \in B.$

Proof:

For $a, b, c \in B$ we have

$$ \begin{aligned} \underline{\Delta}(c.(a \otimes b)) &= \underline{\Delta}(ca \otimes b) \\ &= \sum ca \otimes b_{\underline{1}} \otimes_B (1 \otimes b_{\underline{2}}) \\ &= \sum ca1_{\underline{1}} \otimes b_{\underline{1}}1_{\underline{2}} \otimes b_{\underline{2}} \\ &= \sum c(a1_{\underline{1}}) \otimes b_{\underline{1}}1_{\underline{2}} \otimes b_{\underline{2}} \\ &= c\sum a1_{\underline{1}} \otimes b_{\underline{1}}1_{\underline{2}} \otimes b_{\underline{2}} \\ &= c.\underline{\Delta}(a \otimes b) \end{aligned} $$

$$ \begin{aligned} \underline{\Delta}((1 \otimes b).c) &= \underline{\Delta}\left(\sum c_{\underline{1}} \otimes bc_{\underline{2}}\right) \\ &= \sum \left(c_{\underline{1}} \otimes (bc_{\underline{2}})_{\underline{1}}\right) \otimes_B \left(1 \otimes (bc_{\underline{2}})_{\underline{2}}\right) \\ &= \sum c_{\underline{1}}1_{\underline{1}} \otimes b_{\underline{1}}c_{\underline{21}}1_{\underline{2}} \otimes b_{\underline{2}}c_{\underline{22}} \\ &= \sum c_{\underline{1}\,\underline{1}} \otimes b_{\underline{1}}c_{\underline{1}\,\underline{2}} \otimes b_{\underline{2}}c_{\underline{2}} \end{aligned} $$

(from coassosiativity Δ)

$$ \begin{aligned} \underline{\Delta}(1 \otimes b).c &= \sum (1 \otimes b_{\underline{1}}) \otimes_B (1 \otimes b_{\underline{2}})c \\ &= \sum (1 \otimes b_{\underline{1}}) \otimes_B (c_{\underline{1}} \otimes b_{\underline{2}}c_{\underline{2}}) \qquad (1) \\ &= \sum (1 \otimes b_{\underline{1}}).c_{\underline{1}} \otimes b_{\underline{2}}c_{\underline{2}} \\ &= \sum (1 \otimes b_{\underline{1}})\Delta(c_{\underline{1}}) \otimes b_{\underline{2}}c_{\underline{2}} \\ &= \sum c_{\underline{1}\,\underline{1}} \otimes b_{\underline{1}}c_{\underline{1}\,\underline{2}} \otimes b_{\underline{2}}c_{\underline{2}} \end{aligned} $$

$$ \begin{aligned} (I_{B \otimes_R B} \otimes \underline{\Delta}) \circ \underline{\Delta}(a \otimes b) &= (I_{B \otimes_R B} \otimes \underline{\Delta})\left(\sum a \otimes b_{\underline{1}} \otimes_B 1 \otimes b_{\underline{2}}\right) \\ &= \sum a \otimes b_{\underline{1}} \otimes_B \underline{\Delta}(1 \otimes b_{\underline{2}}) \\ &= \sum a \otimes b_{\underline{1}} \otimes_B (11_{\underline{1}} \otimes b_{\underline{21}}1_{\underline{2}} \otimes b_{\underline{22}}) \\ &= \sum a1_{\underline{11}} \otimes b_{\underline{1}}1_{\underline{12}} \otimes b_{\underline{21}}1_{\underline{2}} \otimes b_{\underline{22}} \\ &= \sum a1_{\underline{1}} \otimes b_{\underline{11}}1_{\underline{21}} \otimes b_{\underline{12}}1_{\underline{22}} \otimes b_{\underline{2}} \\ &= \sum a1_{\underline{1}} \otimes b_{\underline{11}}1_{\underline{21}} \otimes b_{\underline{12}}.1 \otimes b_{\underline{2}} \\ &= \sum a1_{\underline{1}} \otimes b_{\underline{11}}1_{\underline{21}} \otimes b_{\underline{12}} \otimes_B 1 \otimes b_{\underline{2}} \\ &= \sum \underline{\Delta}(a \otimes b_{\underline{1}}) \otimes_B (1 \otimes b_{\underline{2}}) \\ &= (\underline{\Delta} \otimes I_{B \otimes_R B})\sum (a \otimes b_{\underline{1}}) \otimes_B (1 \otimes b_{\underline{2}}) \\ &= (\underline{\Delta} \otimes I_{B \otimes_R B}) \circ \underline{\Delta}(a \otimes b) \end{aligned} $$

Based on Commutative diagram in Figure 3 $\underline{\Delta}$ is a coassosiative weak comultiplication on $B \otimes_R B$. Clearly $\underline{\varepsilon}$ is left B-linear and

$$\begin{aligned}\left(I_{B\otimes_R B}\otimes\underline{\varepsilon}\right)\circ\underline{\Delta}(a\otimes b)&=\left(I_{B\otimes_R B}\otimes\underline{\varepsilon}\right)\left(\sum a\otimes b_{\underline{1}}\otimes_B 1\otimes b_{\underline{2}}\right)\\&=\sum a\otimes b_{\underline{1}}\otimes_B\underline{\varepsilon}\left(1\otimes b_{\underline{2}}\right)\\&=\sum a\otimes b_{\underline{1}}\otimes_B\left(\left(1\otimes b_{\underline{2}}\right).1\right)\\&=\sum a\otimes b_{\underline{1}}\otimes_B 1_{\underline{1}}\otimes\varepsilon\left(b_{\underline{2}}1_{\underline{2}}\right)\\&=\sum a1_{\underline{1}\,\underline{1}}\otimes b_{\underline{1}}1_{\underline{1}\,\underline{2}}\otimes\varepsilon\left(b_{\underline{2}}1_{\underline{2}}\right)\\&=\sum a1_{\underline{1}}\otimes b_{\underline{1}}1_{\underline{2}\,\underline{1}}\otimes\varepsilon\left(b_{\underline{2}}1_{\underline{2}\,\underline{2}}\right)\end{aligned}$$

$=\sum a1_{\underline{1}}\otimes b1_{\underline{2}}=(a\otimes b).1$ (from counital ε)

Analog for

$\underline{\Delta}\circ\left(I_{B\otimes_R B}\otimes\underline{\varepsilon}\right)(a\otimes b)=(a\otimes b).1.$ □

In general properties from $\underline{\Delta}$ and $\underline{\varepsilon}$ are not sufficient to make $B\otimes_R B$ be a coring. Map $\underline{\varepsilon}$ need neither be right B-linear nor $(\underline{\varepsilon}\otimes I)\underline{\Delta}(a\otimes b)=(a\otimes b).1.$ To ensure these properties we have to pose additional conditions on ε and Δ.

Theorem 4.3

Triple (B,μ,Δ) induces a B-coring structure on $B\otimes_R B$ if only if B is a bialgebra with $\varepsilon(ab)=\varepsilon(a)\varepsilon(b)$, for all $a,b\in B$ and $\Delta(1)=1\otimes 1.$

Proof:

$(\Leftarrow)$ Assume B is a bialgebra with $\varepsilon(ab)=\varepsilon(a)\varepsilon(b)$, $a,b\in B$ and $\Delta(1)=1\otimes 1.$

For $a,b,c\in B,$

$(a\otimes b).1=(a\otimes b)\Delta(1)=(a\otimes b)(1\otimes 1)=a\otimes b,$

then $B\otimes_R B$ is unital (B,B)-bimodule and

$$\begin{aligned}\underline{\varepsilon}\left((a\otimes b).c\right)&=\underline{\varepsilon}\left(\sum ac_{\underline{1}}\otimes bc_{\underline{2}}\right)\\&=\sum ac_{\underline{1}}\varepsilon\left(bc_{\underline{2}}\right)\\&=\sum a\varepsilon(b)c_{\underline{1}}\varepsilon\left(c_{\underline{2}}\right)\ (\varepsilon\ \text{properties})\\&=a\varepsilon(b)c\ (\text{counital }\varepsilon)\\&=a1\varepsilon(b)1.c\\&=\underline{\varepsilon}(a\otimes b).c\end{aligned}$$

showing that $\underline{\varepsilon}$ is right B-linear and by Theorem 4.2 so $B\otimes_R B$ is B-coring.

$(\Rightarrow)$ Assume $B\otimes_R B$ is B-coring, then $B\otimes_R B$ is a unital right B-module

$$\begin{aligned}1\otimes 1&=(1\otimes 1).1=(1\otimes 1)\Delta(1)\\&=\sum 11_{\underline{1}}\otimes 11_{\underline{2}}=\sum 1_{\underline{1}}\otimes 1_{\underline{2}}=\Delta(1),\end{aligned}$$

$\underline{\varepsilon}$ is right B-linear so we have

$$\begin{aligned}\underline{\varepsilon}\left((1\otimes a).b\right)&=\left(\underline{\varepsilon}(1\otimes a)\right)b\\&=\sum 11_{\underline{1}}\varepsilon\left(a1_{\underline{2}}\right)b=\varepsilon(a)b\end{aligned}$$

and

$$\underline{\varepsilon}\left((1\otimes a).b\right)=\underline{\varepsilon}\left(\sum b_{\underline{1}}\otimes ab_{\underline{2}}\right)=\sum b_{\underline{1}}\varepsilon\left(ab_{\underline{2}}\right)$$

by applying properties of ε we get

$$\begin{aligned}\sum\varepsilon\left(b_{\underline{1}}\varepsilon\left(ab_{\underline{2}}\right)\right)&=\sum\varepsilon\left(a\varepsilon\left(b_{\underline{1}}\right)b_{\underline{2}}\right)=\varepsilon(ab)\\&=\varepsilon\left(\sum\varepsilon\left(ab_{\underline{1}}\right)b_{\underline{2}}\right)=\varepsilon\left(a\sum\varepsilon\left(b_{\underline{1}}\right)b_{\underline{2}}\right)\\&=\varepsilon\left(\varepsilon(a)b\right)=\varepsilon(a)\varepsilon(b).\end{aligned}$$

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