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Analysis of the meaning of continuity and differentiation: A study on graph problems

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ABSTRACT

Continuity and differentiability are concepts that students must master to effectively learn and solve problems in a variety of contexts. Thus, this study sought to elicit students' interpretations of the relationship between continuity and differentiability, particularly in graphic problems. Through a qualitative approach, this study involved 195 third-year undergraduate students from various Indonesian universities. Ten of them agreed to an in-depth interview for exploration and clarification. Thematic analysis was conducted to deduce patterns from the responses of participants based on the findings. This study discovered three types of meanings that students construct when they solve problems: 1) physical meaning; 2) analytical meaning, and 3) covariational meaning. The three findings could serve as a conceptual framework for future learning processes that emphasize continuity and differentiability. Additionally, our research revealed that Indonesian undergraduate students are unfamiliar with the graphical problems associated with the two concepts. Thus, future research will focus on developing learning strategies that incorporate a variety of representations to improve students' conceptual understanding of calculus concepts

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INTRDUCTION

Numerous researchers in mathematics education have taken an interest in calculus in recent years (Tall, 1991). Calculus is a fundamental area of mathematics that lies at the heart of the mathematics curriculum from middle school to university (Tsamir & Ovodenko, 2013). In our preliminary research, we discovered several studies examining students' difficulties comprehending calculus concepts as a result of their lack of understanding of the topic of function. Additionally, several studies had been conducted on students' conceptual and procedural understanding of the derivative and antiderivative relationships (Garcia-Garcia & Dolores-Flores, 2018). Finally, some research examined the application of calculus concepts to real-world situations or graphs (Ikram, et al., 2020). Calculus's curriculum places a greater emphasis on algebraic representation. As a result, students become accustomed to manipulating algebraic expressions rather than identifying concepts, definitions and performing theorem analysis. Nonetheless, studies examining the relationship between concepts and their various representations, such as the relationship between continuity and differentiability in graph problems, are few and far between. Numerous students frequently overlook the two relationships and attempt to avoid the situation's complications. Additionally, several students graphically misunderstood the statement "if f is differentiable at $x = a$, then f is continuous at $x = a$". Thus, students should understand the relationship between continuity and differentiability because it will aid them in comprehending the calculus concept.

Although many students are capable of successfully solving calculus problems for algebra cases, our review of the literature in the international journal of research in mathematics education revealed a dearth of research on the relationship between continuity and differentiability in algebra. By examining the students, we can provide references for the classroom's calculus learning process, emphasizing the application of the relationship between continuity and differentiability. Teachers can minimize the use of algebra problems to instil calculus concepts (Dibbs, 2019). Numerous expressions involving continuity and differentiability require logic. Students who do not understand the logic behind a statement will have difficulty comprehending and interpreting the proof of a theorem.

Additionally, students' inability to analyze the meaning of a statement's quantification resulted in several errors, such as "if-then," only if," if and if only," "ϵ," dan "$\forall$" (Selden & Selden, 1995). For instance, in the calculus book, there are three statements, and they are: (1) "differentiability implies continuity"; (2) "if f is differentiable at c, then f is continuous at c"; and (3) "if a function is not continuous at a point, then the function is not differentiable." Providing that the students think the other way around, they might realize that continuity does not ensure differentiability, which means that the statement's converse is not correct (Sevimli, 2018a). Additionally, they frequently employed counterexamples to demonstrate that an idea is false.

Several studies indicated some difficulties that students encounter when applying the concept of continuity and differentiability. For example, learners found it difficult to elucidate why $f(x) = |\sin x|$ is not differentiable at $x = \pi$ by employing an analytic approach (Biza & Zachariades, 2010; Mcgowen & Tall, 2010). Additionally, students were less able to coordinate the connection between continuity and differentiability when sketching graphs, causing them to miss the relationship for each interval. It demonstrates that individuals' thought processes are dominated by analytic rather than visual processes. While the majority of students understood that a function is differentiable if the graph contains a tangent, they are unable to describe the process by which the domain value is determined. Finally, they were unaware of the condition that prevents a function from being differentiable based on its graphic representation. As a result, we can investigate the meanings students construct about the relationship between continuity and differentiability through graphic visualization.

In Indonesia's calculus curriculum, the derivative concept is applied to graphs. The majority of students, on the other hand, focused exclusively on the general properties of the function (for example, the shape of the curve or predicting the function's formula) rather than on the derivative's properties (continuity, going up or going down, stationary). They rarely develop their ideas into solutions to graph-related problems. Thus, encouraging them to complete the derivative task will help them develop their skills. This study demonstrates the critical nature of providing students with multiple interpretations of mathematical concepts.

In summary, this study is relevant and necessary because: (1) research on the relationship between continuity and differentiability in graphic problems is still uncommon; (2) the relationship between continuity and differentiability should be the primary focus for students in all countries, including Indonesia; and (3) this study provides information regarding the meanings that the students build as references to teach continuity and differentiability in the future research As such, we propose to address the following research questions in this article:

"What meanings do students construct when they use the relationship between continuity and differentiability on the graphic problems?"

Conceptual Framework of Continuity and Differentiability on the Case of Graph

Individuals are stimulated to construct a reverse process between the derivative graph and its antiderivative by the connection between function and its derivative. When students draw a curve, they encounter the standard procedure, which includes: (1) determining the monotony (whether the curve is ascending or descending) in an interval; (2) determining the curve's highest and lowest points (the vertex); (3) determining the curve's turning point and extreme point; and (4) using the second derivative to determine the curve's concavity. However, the procedures do not improve students' conceptual understanding, such as the meaninglessness of the curve's concavity or whether the curve is going down or up (Berry & Nyman, 2003). Numerous studies examined students' comprehension of the relationship

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between a function and its derivative. In general, their findings indicated that learners encountered difficulties and misconceptions when attempting to interpret the relationship between the derivative graph and its antiderivative, including the extreme point, horizontal tangent at a point, and the symbol for the second derivative. The results provide a preliminary overview of how difficult it is for students to sketch the graph of a function.

The difference in students' preferences also has an effect on how they interpret graphs (Haciomeroglu et al., 2010). Visual thinkers frequently determine the graph of the function by observing the slope change at the derivative curve but fail to interpret the shift near the graph's vertical tangent based on the function derivative graph. Analytic thinkers are prone to relate problems to their algebraic expressions, but they have difficulty associating continuity and differentiability. Additionally, propose two distinct thinking models for children when they construct a graph, namely a process dominated by (1) algebraic thinkers and (2) interval thinkers. The distinction between the two models is determined by an awareness of the meanings and relationships between the elements in the problems encountered, the presence of knowledge reconstruction to create new structures, and the ability to rearrange existing knowledge. It demonstrates the importance of synthesizing students' thought processes to supplement students' comprehension of calculus concepts.

Continuity and differentiability are two essential concepts in calculus that involve formal definition limits and various theorems (Craig Swinyard & Sean Larsen, 2012). However, the students just seem to recognize the limit as a function at a point or as objects to solve problems formally (Fernández-Plaza & Simpson, 2016). Thus, the connection between continuity and differentiability is less coherent for them. Some textbooks have outlined theorems related to the concepts, for example: (1) a function is continuous at a point if $\lim_{x \to c} f(x)$ and $f(c)$ (c is an element of the domain of f) exist, and $\lim_{x \to c} f(x) = f(c)$; (2) f is differentiable at c if $\lim_{h \to 0} \frac{f(x+c)-f(x)}{h}$ exist; and (3) if f is differentiable at c, then f is continuous at c (for every function $f\colon \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$). On the other hand, if a function is discontinuous at a point (as a result of a jump, for example), it is not differentiable at that point (Sevimli, 2018a). There is a case in which a function is continuous but not differentiable (for example, the vertical tangent or when the function has a high degree of curve). Students should analyze situations in which continuity does not necessarily imply differentiability.

METHOD

To begin, we will discuss the research context or the study's current state. Following that, we discussed the participants, task designs, data collection procedures, and analysis of the data. The study is exploratory in nature and focuses on the meanings students construct when they apply the relationship between continuity and differentiability to graph problems.

Context of the study

This study is a continuation of the research conducted by Ikram et al. (2020) on the reasoning displayed by students when sketching graphs involving the derivative concept. The study hypothesized that discrepancies exist when students draw a graph with continuity and differentiability. As a result, the study concentrated on the issue of whether a function is continuous and differentiable at a point. Following that, it was used to address the research question regarding the interpretations students make in order to obtain answers. Thus, students are likely to conduct various analyses to ascertain the correct answers, which will involve the curve behaviour, where the function is decreasing or increasing, in order to solve the problem analytically.

Participants

The study recruited 145 third-year undergraduate students from a variety of universities. Their ages ranged between 19 and 20 years. The students had 45 minutes to complete the problems in Google Form. We contacted our colleagues at numerous colleges to ensure that their students successfully completed the assigned task. All students who took part in the study were currently enrolled in an advanced calculus course. According to their universities' calculus curricula, the majority of students were taught how to solve problems procedurally (For instance, determining the derivative of a function

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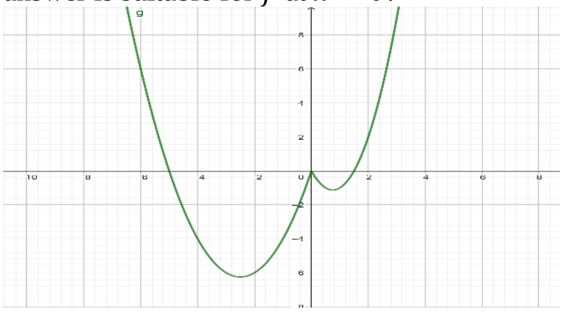
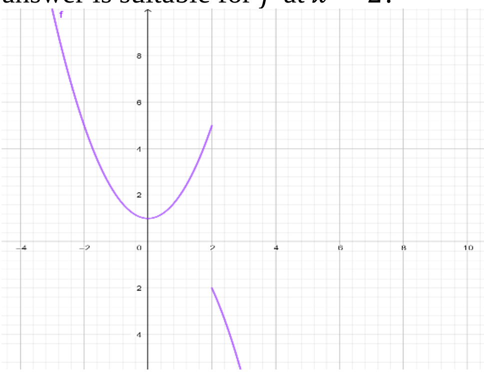
and draw a graph with the known function formula). As a result, they had few opportunities to apply calculus concepts to graph problems. Numerous students struggled with analyzing the relationship between continuity and differentiability. Additionally, some students analyzed continuity alone, disregarding differentiability. The results section discusses the students' responses.

The instrument for collecting data

A questionnaire and an interview protocol were used to collect data. Three tasks were included in the questionnaire, each of which contains references to continuity and differentiability. The assignment is as follows.

Table 1. Task Development

Task	Description
Task #1 The following figure is the graph of $f(x)$. Which answer is suitable for f at $x = 0$?  ○ f is continuous but is not differentiable ○ f is discontinuous but is differentiable ○ f is continuous and differentiable ○ f is discontinuous and not differentiable	The graph of f given has a kink at $x = 0$. The results expected are: 1. Students interpret the curve behaviour of f near $x = 0$, for example, f is increasing if $x < 0$, and is decreasing immediately if $x > 0$ 2. Students use the limit definition to obtain the conclusion about continuity and differentiability. 3. Students apply the relation between continuity and differentiability, but they need to consider the sharp corner of f at $x = 0$ 4. Students construct their ideas by employing various representations to find solutions
Task #2 The following figure is the graph of $f(x)$. Which answer is suitable for f at $x = 2$?  ○ f is continuous but is not differentiable ○ f is discontinuous but is differentiable ○ f is continuous and differentiable ○ f is discontinuous and not differentiable	The graph of f has a jump at $x = 2$. It is a piecewise function. The expected results are: 1. Students interpret the behaviour of the curve f around $x = 2$. 2. Students employ the definition of limit, continuity, and differentiability to conclude. 3. Students use the relation of continuity and differentiability, which is if f is discontinuous, then f is not differentiable. 4. Students develop their ideas by utilizing numerous representations to obtain answers.
Task #3 The following figure is the graph of $f(x)$. Which answer is suitable for f at $x = 1$?	The graph of f has a jump at $x = 1$. The function is a piecewise function. The expected results are: 1. Students interpret the curve behaviour of f around $x = 1$. 2. Students use the definition of limit, continuity, and differentiability to make conclusions

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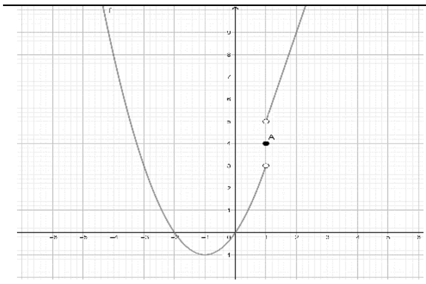
Task	Description
 ○ f is continuous but is not differentiable ○ f is discontinuous but is differentiable ○ f is continuous and differentiable ○ f is discontinuous and not differentiable	3. Students use the relation of continuity and differentiability. 4. Students construct their ideas by using various representations to obtain answers.

We formulated the task by some considerations. To begin, graph aspects of concepts of continuity and differentiability are critical for students' conceptual understanding and perspective. Second, we discovered that the majority of students concentrated exclusively on problems involving algebraic representations. One involving graphical representations was rarely provided. It reveals that students' experience and the inability to apply the algorithm to obtain solutions led us to believe that students' thinking was challenged in order to interpret the graph of f and to apply concepts of continuity and differentiability. Finally, each task includes graphical representations. Ten students participated in a follow-up interview to clarify their responses based on the solution obtained. We conducted interviews with students who provided an interesting response to the questionnaire. We showed their original solutions during the interview. They were tasked with providing detailed justifications for the concepts they wrote. The interview lasted approximately 10-15 minutes and was recorded and transcribed using an audio recorder.

Data analysis

We analyzed the data using thematic analysis. It deduces patterns (themes) from respondents' responses using the instruments provided (Clarke & Braun, 2013). We chose this method for a variety of reasons. Firstly, there is no prior framework for examining the meanings of continuity and differentiability, which could serve as a contribution of this study. Secondly, the method is adaptable and could be used to address the research question, specifically regarding the meanings students construct when applying the relationship between continuity and differentiability to the graph problem. Following that, we can use the methods to analyze a large data set.

We discovered patterns as well as unique naming using the thematic analysis method. The contribution of this study is significant through thematic analysis, as studies utilizing the method in the field of mathematics education are still uncommon. To accomplish our objectives, we adapted the phase proposed by Braun & Clarke (2013). There were six stages, which are as follows:

Phase 1: *familiarizing yourself with your data*. We read students' work and interview transcripts during this phase. It assisted us in identifying the concepts, verbal expressions, and thought processes revealed during the interview.

Phase 2: *generating initial code*. We developed preliminary codes based on our general reading of the transcript and the conceptual framework used. To explicate the relationship between continuity and differentiability, we examined participants' verbal expressions while solving problems, such as the following interview excerpt.

Interviewer	*What is your answer in Task#1?*
Dwi	*f is continuous at $x = 0$, but it does not have a derivative*
Interview	*How do you come to that conclusion?*
Dwi	***The graph seemed continuous, without hole, jump, and distance.** Thus, I conclude that f is continuous at $x = 0$. However, **the curve also has a high***

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__degree of the curve__ at $x = 0$. It means that f does not have a derivative at the point or is not differentiable at $x = 0$.

Interview *What is the solution to Task#1?*
Feri *f is continuous, but it is not differentiable?*
Interview *What do you think?*
Feri *f is continuous at $x = 0$ because the value of limit as x approach 0 exists and equals to $f(0)$. However, it is not differentiable at $x = 0$ because the value of its derivative or*

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

The values from the right and the left are different
Thus, __I utilize the limit concept to conclude.__

We generated codes based on the bolded verbal expressions. For instance, S2 employed curve behaviour to solve problems, whereas S3 relied on analytical identification. We created eight codes in this phase to represent the meanings of the relationship between continuity and differentiability.

Phase 3: *Looking for themes*. In this phase, we made, determined, and modified codes to understand relationships and formed themes. We grouped codes with the same meanings; for example, the code "f curve is continuous and has a sharp corner at $x = 0$", and the code "f curve has a kink at $x = 0$". In this case, we utilized the theme: physical meaning because the codes' main characteristics showed that students concluded by using the behaviour around the point in question.

There was also the theme of analytical meaning. We grouped the code "The $f(0)$ as well as the right-hand and left-hand limits of $f(x)$ as x approach zero exists, but $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ have different values when it approaches zero from the left and the right" and "continuous because the limit value exists, but is not differentiable because its derivative values are different." It implied that students employ the analytic properties to investigate the problems.

We also used the graphical meaning to form other codes. For example, the code "derivative is the gradient of the curve tangent," or "the curve f is increasing as x approach 0 from the left and the right, and its derivative value decreases, decreasing". We need to consider it as visual meaning because they use changes near x to make conclusions.

Phase 4: *reviewing themes.* We discussed the relationship between the themes and our data repeatedly during this phase. The process would become a stage obtaining the relationship between continuity and differentiability that students construct.

The code was regarded as physical meaning in the case of covariational meaning. Nonetheless, our team discussions concluded that we must develop a new theme. We believed it has its answer patterns through the use of curve behaviour and visual representations of derivatives.

Phase 5: *defining and naming themes*. We defined and labelled the three ways in which students develop meanings for the connection between continuity and differentiability: physical meaning, analytical meaning, and visual meaning.

Phase 6: **Producing the report**. In this phase, we wrote the final reports of our research results. Besides, we carried out data triangulation to enhance the objectivity of our findings. We improved trustworthiness by discussing our research results with experts in mathematics education to achieve mutual agreements. We ensured that the data obtained is accurate and complete by administering the task in written form and transcribed every interview immediately after recording it. There was also a validation of the coding process and recoding of different categories through discussion with several mathematics education experts.

RESULTS AND DISCUSSIONS

In this section, we present our results from 195 students participating in the study. We specifically highlight students' responses in Task#1, where 156 out of 195 students answered that f is continuous and differentiable at $x = 0$. We collected several answers as to why they choose the solutions, and Table 2 presents the summary.

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Tabel 2 Students' Answers in Task#1

Answer	Response	Number of Students
f is continuous and differentiable at $x = 0$	Meets the conditions of continuity and differentiability	32
	Continuous because both of the limits as x approach 0 are 0. It is differentiable due to $f'(x) > 0$ from the left of $x = 0$ and $f'(x) < 0$ from the right of $x = 0$	64
	The graph is continuous, which means that f has a derivative at $x = 0$	60
f is continuous but is not differentiable at $x = 0$	Ignoring the differentiability's properties	19
	The curve f is continuous, but have a sharp corner at a point	8
	Continuous because $\lim_{x\to 0} f(x)$ exists. It is not differentiable because $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ did not exist	7
	The gradient of f at $x = 0$ is different	5

Students' data who provide incorrect responses in our study become the participants of subsequent research. Following that, 39 students out of 156 responded that f is continuous but not differentiable at $x = 0$. They cite a variety of reasons. This section is the primary focus of our investigation in order to determine the answer to our research question. Nineteen of them tend to ignore the properties of differentiability in order to solve the problems. They reasoned that the graph of f is connected between the intervals $x > 0$ and $x < 0$. Additionally, they stated that the curve near $x = 0$ is devoid of jumps and holes. This indicates that they overlooked the differentiability properties in order to reach conclusions. Additionally, we interviewed them to clarify their responses. The excerpt is as follows.

Interviewer	*Why did you answer that f is continuous but is not differentiable at $x = 0$?*
Wika	*The graph of f at $x = 0$ has no interference, for example, no hole, jump, or no cuts*
Interviewer	*How about the condition of its differentiability?*
Wika	*If it does not have things such as holes or jumps, then it is differentiable at $x = 0$ or* ***when f is continuous, but I do not know the reason why it does not have a derivative.***

The excerpt shows that students overlooked the properties of differentiability because they believe continuity results in differentiability. In other words, they solved the problem through the application of procedural knowledge.

Eight out of forty-three students correctly stated that f is continuous but not differentiable at $x = 0$. The reason was the curve behaviour of f, which is continuous and form breaks. It shows that they recognized that if the curve of f has a break or kink, it does not have a derivative value. Yaqin, one of the participants, answered the question by utilizing the gradient of the curve tangent and predicting the graph's function formula. He stated that (1) the slope of f is different from the left and the right of $x = 0$ (1 and -1), indicating that the function does not have a derivative at $x = 0$; and (2) the graph of f contains two curves, indicating that it is a piecewise function. At $x = 0$, the function has breaks, indicating that it has no value. We interviewed two students to ascertain the specific causes of these difficulties, and the following are excerpts from the interviews.

Interviewer	*Why did you say that f is continuous but is not differentiable at $x = 0$?*
Dwi	*The graph seems to be continuous without hole, jump, or distance, so I conclude that f is continuous at $x = 0$. However, it has a sharp corner at $x = 0$. It implied that f does not have a derivative at the point or is not differentiable at $x = 0$.*

Students concentrated on the curve behaviour of f in order to conclude its differentiability and continuity. This indicated that they interpreted the situation as having a physical meaning. They

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reasoned that the continuous curve demonstrates continuity properties (the curve of f which does not have a jump, hole, or asymptote). The existence of a sharp corner in the curve demonstrates the differentiability's properties. We believe that their observation of curve behaviours was critical and warranted further discussion in this study. Students' constructed meanings provide valuable insights and references for the classroom teaching process.

Apart from using the curve behaviour of f to answer the question, 7 out of 39 participants used the analytical properties of continuity and differentiability by employing the formal definitions. For instance, Ristia explained that the graph of f is continuous because it does not have a jump or smooth. She also added that its curve meets the conditions of continuity in function, that is, (1) the right-hand and left-hand is defined or $\lim_{x\to 0} f(x)$ exists; (2) the function value at $x = 0$ exists; and (3) the value of its limit and its function is the same or $\lim_{x\to 0} f(x) = f(0)$. Next, to identify its differentiability, the participant used the definition of derivative, that is, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ (examining when $x \to 0^-$ and $x \to 0^+$). Moreover, the participant also employs the ideas of the previous problem (e.g., $f(x) = |x|$) to conclude. It implied that students used analytical meaning to explain the relationship between continuity and differentiability. We deem this necessary to be explored as a reference in the learning process. The results of our interviews with the participants who fall into this category are as follows.

Interviewer	*How did you realize that f is continuous but is not differentiable at $x = 0$?*
Dwi	*Based on the function graph, it is continuous because it does not have a jump or smooth. Formally, it met the conditions of continuity of a function, which are: 1. The limit exists (the right-hand and the left-hand limit is the same); 2 The function value at $x = 0$ exists. Therefore, it can be concluded that f is continuous. The function, however, is not differentiable because:* *The values of "$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$" are different*

The remaining students (4 out of 39), for example, Feri, provided straightforward explanations for why f is continuous but not differentiable at $x = 0$. They used a counterexample to demonstrate the relationship between continuity and differentiability; that is, there is a condition in which f is continuous at $x = a$ but is not differentiable at the point. He defined the continuity condition verbally as follows: when x approaches 0 from the left and right, the value of the function $(f(0))$ will be the same, indicating that f is continuous at $x = 0$. In terms of differentiability, he was familiar with the definitions of derivatives used to represent the gradient of a curve's tangent. Thus, he contended that the slope of f is different on the left and right and that the function is not differentiable at $x = 0$. We classify the meanings he constructs as covariational meanings because he interprets the curve of f using the relationship between two quantities. Our findings should serve as a guide for students as they attempt to grasp the relationship between continuity and differentiability. The following is an excerpt from our interview.

Interview	*What do you think about your answer saying that f is continuous but is not differentiable at $x = 0$?*
Dwi	⋯ [pause for a moment]⋯ when x approach 0, approach 0, approach 0, then the function's values are the same from the left and the right. Next, the function value is the same. Thus, f is continuous. Then, this, its gradient increase, increasing from the left of 0. From the left, the gradient also increases, but their values are different. Then, the curve f is not differentiable at $x = 0$,

According to our findings, when students solved graphic problems involving the relationship between continuity and differentiability, they developed three meanings. There are three types of meaning: physical meaning, analytical meaning, and covariational meaning. We will discuss each of them in greater detail in the discussion section.

We discovered some unexpected outcomes. One hundred fifty-six participants provide incorrect responses. We determined that additional investigation is necessary for future research. Additionally,

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19 students occasionally guessed the solution without considering the requirement of differentiability, necessitating a subsequent investigation into the situation.

We discuss our findings in this section in relation to the relationship between continuity and differentiability that students construct in the graphic problem. The first is physical meaning, in which students used a particular graph behaviour to solve problems. Some participants developed analytical or covariational meanings. Learners who are developing the former required symbols or signs to translate the problems' situations. In the latter case, students constructed meaning by utilizing two related quantities concurrently. We will discuss the definitions in greater detail, as well as their implications for practice and calculus curriculum in the classroom.

Students strove to explore the function formula in the graphic context to conclude the continuity and differentiability of a point. They did not, however, understand the implied meanings of the function's graph, such as the rate of change of the value of x toward y, the increase and decrease of the f curve, its gradient, or a particular condition of the curve. According to our findings, the majority of students failed to interpret the propositions of the continuity and differentiability theorems (156 out of 195). They reasoned that if f is differentiable at c, it must also be continuous at c. However, the participants were unaware that the proposition's converse is incorrect. Our findings corroborated Viholainen's (2008) findings that some students believed that the condition of continuity resulted in the requirement for differentiability being met. Students' difficulties with the proposition can be minimized by encouraging them to think reversibly about the proposition statements, to use counterexamples, to control their mental schemas, and to create a visual schema (concept map) based on the relationships between the concepts presented (Ikram, et al., 2020; Sevimli, 2018a).

When students identified the continuity and differentiability in the graphic problems, most participants observed the behaviour and the shape of the graph of f. For instance, in Task #1, 8 out of 195 students interpreted the increase or decrease of the curve f around $x = 0$, and predicted the function formula of the curve f as a piecewise-defined function. Based on their answers, they realize that the continuity of f at $x = 0$ is caused by its discontinuous curve (in the form of jump, hole, or vertical asymptote). In terms of differentiability, they thought that the f curve has a sharp corner at $x = 0$ and similar with the function $f(x) = |x|$ which is not differentiable at $x = 0$. It implied that participants used physical meaning to answer the questions. The situation is called "graph of f have a sharp corner," which means that its left-sided and right-sided derivatives are different (Sevimli, 2018b). When the problems were extended, they were also aware that if the graph of f has a vertical tangent, then it is continuous, but is not differentiable at the point. Students' flow of thinking which use physical meaning tended to carry out a decomposition of the problem or solve it part by part (for example, students solve the continuity and then continue to its differentiability). They separated the problems into sub-section based on the sequence of the problems and proceeded by analyzing each sub-section separately (Rich et al., 2019).

There was also another finding showing that some students construct the relation between continuity and differentiability by using limit as a basis, for example, if f is continuous at $x = 0$ is translated to $\lim_{x \to 0} f(x)$ exists, and f is differentiable at $x = 0$ translated to $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ is defined. It shows thinking of symbols or notations influences the students to solve problems and their mental schemas activate the idea about the definitions of continuity and derivative (García-García & Dolores-Flores, 2019). In the case of differentiability, our findings are consistent with a study by Park (2015) showing that students were aware of the meanings of differentiability $f'(0)$ represented as $\frac{f(x)-f(0)}{x-0}$ and add the symbol of $\lim_{x \to 0}$ in $\frac{f(x)-f(0)}{x-0}$ represented as the tangent in the graph at $x = 0$. Our results also supported the study by Sánchez-Matamoros et al. (2014), revealing that the use of limit (*limit of the difference quotient*) to the meanings of derivative as tangent are essential parts of students' analysis in identifying and interpreting the elements of the problem. Moreover, students thinking tend to be dominated by *symbol-sense*, causing every information in the task to be expressed symbolically (Zehavi, 2004). It shows that the use of *analytical meaning* by utilizing symbols or notation become the important part of our research findings.

Our subsequent findings are related to students' covariational perspectives, in which they viewed problems through the lens of proportion, rate of change, and the function's two variables. Learners described the relationship between two changing quantities, such as: (1) when x approaches zero from

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the left and right, the value of the function approaches zero as well; (2) the gradients of the f curve from the left and right are different, indicating that f is not differentiable at $x = 0$. In this instance, students engaged in cognitive activities by simultaneously coordinating changes and variations in change in quantity (Thompson & Carlson, 2017). Additionally, Sevimli's (2018b) observation of the relationship between continuity and differentiability is very similar to the covariational meaning. Students occasionally expressed the relationship between quantities when interpreting the graph in the study. They did not, however, reach a general conclusion. According to Kertil et al. (2019)'s findings, students' ability to synthesize two quantities demonstrates a high level of thinking. Students who use covariational thinking in their problem-solving are uncommon.

Students' prior experiences have an effect on their three meanings of the relationship between continuity and differentiability. They create new knowledge based on prior knowledge in order to solve problems. Learners who correctly answered could recall their prior knowledge, whereas those with incorrect solutions required guidance to understand the mathematical relationship between the problems. Additionally, the role of teachers in the learning process is a factor that influences the use of prior knowledge (Mcgowen & Tall, 2013). Thus, our findings are strongly related to the integration of students' experiences and classroom lessons, particularly regarding continuity and differentiability problems.

The implication of the relationship between continuity and differentiability in the classroom

We conducted a cursory review of the calculus textbooks used in Indonesia to ascertain their understanding of the importance of continuity and sociability. We were aware that the majority of students encountered procedural problems more frequently than graphical and analytical ones, and we encourage students to seek out a counterexample. The authors of the textbooks only used how the differentiability of $f(x) = |x|$ at $x = 0$ is, which is presented analytically and graphically. Nonetheless, there was no explicit emphasis in its practice on the importance of the relationship between continuity and differentiability. Furthermore, none of the authors examined the impact of textbooks on students' perspectives on unfamiliar problems. As a result, we were unsurprised that the majority of learners were unable to distinguish between when the function f is differentiable and the symbols in the graph of f that cause it to be differentiable.

We conclude that the three meanings developed by students should be used to help students gain a complete understanding. This means that learners' conceptual and procedural comprehension may improve (Scheibling-Sève et al., 2020). Additionally, our findings indicate that the three meanings they constructed should be taken into account as they work to develop a more productive meaning of continuity and differentiability. The difficulties students encountered in comprehending the two concepts should be investigated further. As educators, we expect students to: (1) learn how to determine whether a function is continuous or differentiable using various representations; (2) understand the situations and conditions under which the f curve is continuous but not differentiable; (3) apply continuity and differentiability theorems that require knowledge of propositions; and (4) explicitly pay more attention to physical meaning, analytic meaning, and symbolic meaning. To help students develop a strong conceptual understanding of the two concepts, we suggest that teachers provide numerous opportunities for students to construct their ideas productively in the calculus context, rather than focusing exclusively on specific types of problems.

CONCLUSSION

The purpose of our research was to determine the meanings that students construct when they applied the relationship between continuity and differentiability to graph problems. We discovered three types of meanings that students develop through our thematic analysis: physical meaning, analytical meaning, and covariational meaning. Our findings can serve as a foundational theoretical framework for future research on continuity and differentiability in calculus, particularly in learning. The majority of the meanings constructed by students were influenced by the textbooks they used. Finally, our findings indicate that the three meanings can be used to summarize students' reflections on their thinking while completing the graphic task involving continuity and differentiability.

In general, our findings indicate that students' perspectives on problems and efforts to develop their ideas may aid them in analyzing the graphical problem, such as how their beliefs about the

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continuity and differentiability of a function are represented visually. Due to the graphic problem's centrality in the calculus curriculum, students may underestimate the value of studying the problem in its alternative contexts as a new challenge. On the contrary, it may be necessary to explicitly provide students with multiple representations in the classroom. Explicitly, the meanings that students construct should be developed through problem-solving with a variety of representations. Additionally, our findings are contextualized by the fact that the majority of students did not understand the relationship between continuity and differentiability. As a suggestion for future research, the difficulties could be minimized by incorporating the three meanings discovered in future learning.

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